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The Absolute Measurement of Resistance by the Method of Albert Campbell

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XII—The Absolute Measurement of Resistance by the Method of Albert Campbell

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1—Introduction

Throughout the last 25 years nearly all precision measurements of electrical resistance have been expressed in terms of the International Ohm, a unit which was defined by the London Conference of 1908 (Glazebrook, 1922–3) as follows : " The International Ohm is the resistance offered to an unvarying electric current by a column of mercury at the temperature of melting ice, 14·4521 grammes in mass, of a constant cross-sectional area, and of a length of 106·300 centimetres". Any measurement in terms of this unit is dependent on experiments in which a glass tube, the dimensions of which have been accurately measured, is filled with pure mercury, the weight of which is also measured, and then, the tube being immersed in melting ice, the resistance of the mercury column is compared with that of a standard resistor, which is thereby calibrated in International Ohms. Such measurements are by no means easy, and although the International Unit is specified to 1 part in 10^5, it has not been found possible to get this accuracy with a single tube of mercury, and even when about ten tubes are used and a mean result taken, the accuracy is usually no better than 2 parts in 10^5. The results obtained by the various national laboratories have from time to time differed by about 3 parts in 10^5, and it follows that results expressed in International Ohms are always subject to uncertainties of this order.

It is well known that the International Ohm as defined above was intended to have a value nearly equal to 10^9 C.G.S. electromagnetic units of resistance, this value being regarded as the ohm, as distinct from the International Ohm. Measurements in terms of ohms may be made by any of the so-called absolute electromagnetic methods of resistance measurement, and in 1912 F. E. Smith (now Sir Frank Smith) showed that measurements could be made by the Lorenz method (Smith, 1914) with an accuracy of about 2 parts in 10^5. Shortly afterwards Grüneisen and Giebe at the Physikalisch-Technische Reichsanstalt made measurements by another method, involving an alternating current bridge network and also Maxwell's commutator method of measurement of capacitance. Their investigation also gave an accuracy estimated at about 2 parts in 10^5, and although the results in ohms appeared to differ from Smith's by 4 parts in 10^5, it became clear that the ohm could be realized with an accuracy at least of the same order as that obtained for the International Ohm. The question then arose: Is there any justification for the retention of both units, one for purely scientific work and one for industrial purposes? The matter was considered by the appropriate Committees of the International Bureau of Weights and Measures, and it was ultimately decided to abandon the International Units and to adopt for international use an absolute electromagnetic system. The change is due to take place in 1940, and meanwhile it is of importance that absolute measurements shall be made by as many methods as possible, in order that the relations between the new units, the old units, and standards of all related quantities (including those of inductance, capacitance, length, and time or frequency) may be established with the greatest possible accuracy. The present investigation was undertaken with that end in view.

The method (Campbell, 1925) is a development of one due to Albert Campbell (and published in 1925) in which resistance is determined in terms of a mutual inductance and of the frequency of an alternating current. The value of the mutual inductance is obtained by reference to a primary standard the value of which is calculable from its linear dimensions, so that the measurements are absolute in the sense that they are made by reference to the standards of length and time.

The construction of a primary standard of mutual inductance is of course an essential part of the complete investigation, but such a standard was already available. Its construction and calibration have already been described (Campbell, 1907, 1912; Dye and Hartshorn, 1927), and in the present paper only the electrical measurements will be discussed in any detail. The linear dimensions of the standard were re-determined and the value of its mutual inductance recalculated specially for the present investigation.

2—Principle of the Method

If an alternating current be passed through a resistor (r, fig. 1a) and the primary winding of a mutual inductor (M_1, fig. 1a) in series, then the potential difference across the resistor will be proportional to its resistance, and the electromotive force generated in the secondary winding of the inductor will be proportional to the frequency of the current and to the inductance. If these voltages can be compared, then a relation between resistance, frequency, and inductance can be established. Direct comparison is not possible, since the voltages differ in phase by 90°, and various devices have been suggested to circumvent this difficulty. Albert Campbell used a second mutual inductor (M_2, fig. 1a) to rotate the phase of the induced e.m.f. through another 90°, thus producing an e.m.f. dependent on frequency and inductance and in opposite phase to the potential difference across this resistor, so that comparison becomes possible by balancing one against the other. The resulting circuit is shown in fig. 1a and the corresponding vector diagram in fig. 1b. Since the combination of the two inductors cannot produce a phase rotation of exactly 180°, owing to the self inductance of the secondary windings, it is necessary, in order to secure perfect balance, to add to the two voltages already obtained a third (small) voltage taken from the potential terminals of a resistance (S, fig. 1*a*) included in the secondary or linking circuit. By adjusting M_2 and S a perfect balance of the galvanometer G (fig. 1*a*) can be obtained, the conditions necessary for this being contained in the following equations

$$\omega^2 M_1 M_2 = Rr \quad \ldots \quad (2.1)$$

$$M_1 S = Lr, \quad \ldots \quad (2.2)$$

the frequency of the alternating current being $\omega/2\pi$ cycles per second, and all the other quantities being measured in any system of absolute electromagnetic units.

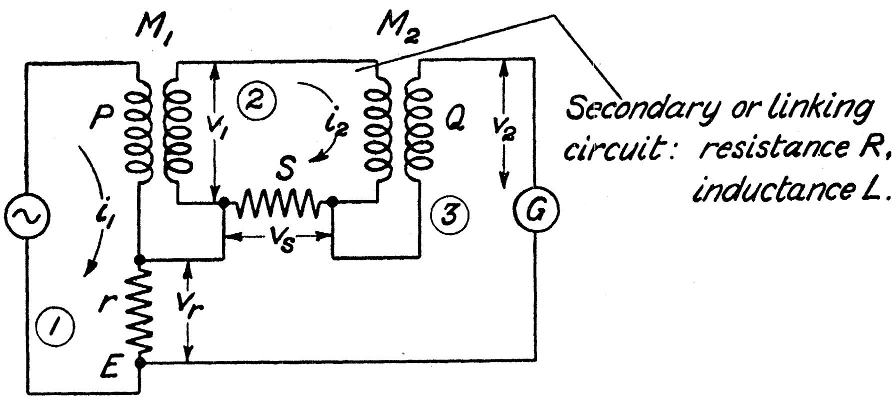

FIG. 1—(*a*) Campbell network in it simplest form.

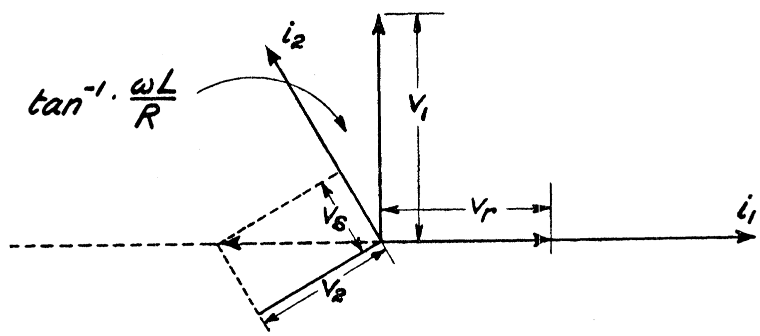

FIG. 1.—(*b*) Vector diagram.

If now M_1 and M_2 are measured in terms of a standard the value of which is calculated from its linear dimensions, and the frequency is measured by reference to a standard clock, the product R*r* can be determined, using equation 2.1. The ratio R/*r* can then be determined by a simple comparison, whence the determination of the value of each resistance in absolute measure can be made.

The system of units chosen will depend on the unit of length (l) adopted for the measurement of the linear dimensions of the standard of mutual inductance, and the value assigned to the coefficient (μ) representing the permeability of free space in the formula for mutual inductance. We shall here employ the henry as unit of inductance (*i.e.*, $l = 1$ cm. and $\mu = 10^7$ or $l = 1$ m. and $\mu = 10^9$) and the values of R and r will therefore be given in terms of the corresponding practical unit, the ohm.

Since our chief object was to determine the ratio of the ohm so defined, and the value now accepted for the International Ohm, the most convenient procedure was to determine the product Rr in ohms as explained above, then to measure both R and r in International Ohms by reference to the Laboratory standards of resistance, and to deduce the required ratio from the two values for Rr.

3—Practical Limitations

Certain assumptions are implicit in the derivation of the conditions of balance given in equations 2.1 and 2.2. These may be stated in the following way :

(*a*) The whole inductance of the network, excluding that of the source of current and detector of balance, is supposed concentrated in the four coils forming the primary and secondary windings of the mutual inductors M_1 and M_2.

(*b*) The only currents flowing in the system are those flowing along the actual wires forming the network ; in other words, eddy currents and displacement currents are ignored.

It is necessary, therefore, to consider how an actual network may depart from these ideal conditions and to determine quantitatively the effect of any departures on measurements made with the network. Such departures will, in general, lead to corrections of the first order of smallness, so that it is permissible to consider the corrections independently of one another, and to deduce any final correcting term as the sum of a number of small terms.

(*a*) *Stray Inductance*

As far as assumption (*a*) is concerned, every wire in the network will possess a finite self inductance, and there will also be a finite mutual inductance between most of the wires taken in pairs. In order to approximate as closely as possible to the ideal conditions in setting up the actual circuit, all the resistors were wound so as to possess as small an inductance as possible, and the various connecting leads were disposed in such a manner that their inductances were reduced to a minimum ; the go and return leads to each component of the network were well twisted together, and the different pairs were separated so as to minimize undesirable mutual inductances. Campbell in his original paper considered the most important of the corrections arising from stray inductances. They are l, the self inductance of the resistance coil r, and m the mutual inductance between the coils P and Q (fig. 1*a*). The self inductance λ, associated with the resistance S, is also of importance. It is to be noted that there are four leads connected to each of the resistance coils r and S, two of which carry current (the current leads) and two (the potential leads) which do not when the condition of balance is reached. The potential difference introduced into the galvanometer circuit by these resistors will obviously depend not only on the resistance and self inductance of the winding of the coil, but also on the mutual inductance between the current and potential leads. The resultant effect is taken into account by measuring l and λ, the effective self inductance of the coils r and S respectively defined by vector equations of the form

$$V = I\,(r + j\omega l) \qquad (3.1)$$

where V is the potential difference between the terminals of potential leads, and I is the current flowing through the coil by way of the current leads. When these small stray inductances are taken into account, the equations* become

$$\omega^2\,[M_1(M_2 + \lambda) + L(l + m)] = Rr \qquad (3.2)$$

$$M_1S + R\,(l + m) + rL = 0 \qquad (3.3)$$

The mutual inductances of all the other connecting leads were taken into account by regarding them as modifying slightly the values of M_1, M_2, and m. Thus, consider the circuit as composed of three meshes, denoted by (1), (2), and (3) in fig. 1*a*. The mutual inductance between the two complete meshes (1) and (2) was taken as M_1; that between meshes (2) and (3), excluding the part due to the current and potential leads of S, which has already been specified as included in λ, was taken as M_2, and that between (1) and (3) excluding the part due to the current and potential leads of r, which has already been specified as included in l, was taken as m. When these values are applied to equations 3.2 and 3.3, all the stray inductances are correctly allowed for. The methods adopted for measuring M_1 and M_2 so defined, and for reducing m to zero are described in later sections.

(*b*) *Eddy Currents and Displacement Currents*

Departures from our second ideal condition are of two kinds, which may be considered as due to the existence of eddy currents in the metallic components of the apparatus, and displacement currents in the dielectric components, in addition to the main current along the various wires. The alternating magnetic fields of the various coils must generate eddy currents in all neighbouring metallic objects, and such currents affect the values of the potential difference between the various points of the network. Further, the network possesses not only a magnetic field but also an electric field, and the alternations of this field constitute displacement or capacity currents in the dielectrics, and these currents also affect the potential differences between the several points of the network and therefore the conditions of balance. Thus, in order to make the diagram of fig. 1*a* completely represent the network in practice, it would, strictly speaking, be necessary to add additional closed circuits

* The signs of the correcting terms given here differ from those in CAMPBELL's original paper. Our convention as to sign is discussed later.

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to represent eddy currents and condensers to represent the paths of the displacement currents. Consider, for example, a coil in any one current path infinitely remote from all other objects. The total current in the path enters one terminal of the coil and leaves the other, and this total current is constant in every part of the path, but in the interior of the coil this current consists partly of conduction current, including eddy current, in the windings, and partly of capacity or displacement current between the turns. The behaviour of such a coil in the network is not accurately characterized by its resistance R_0 and inductance L_0 alone, and the usual simple equation for the voltage across the coil V in terms of the current, I_0, through the winding

$$V = (R_0 + jL_0\omega)\, I_0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3.4)$$

can be regarded only as an approximation, since the value of I_0 is indefinite. If, however, we confine our attention to the terminal current I, at any frequency $\omega/2\pi$, we may write

$$V = [R_0 + \Delta R + j\omega\,(L_0 + \Delta L)]\, I, \quad . \quad . \quad . \quad . \quad . \quad . \quad (3.5)$$

in which, by analogy with 3.4, ΔR and ΔL may be regarded as correcting terms representing the effects of capacity and eddy currents at the frequency $\omega/2\pi$ on the resistance and inductance of the coil respectively, and this equation will represent the properties of the coil accurately at the specified frequency so far as any circuit connected to its terminals is concerned. $R_0 + \Delta R = R$ is called the effective resistance of the coil and $L_0 + \Delta L = L$ its effective self inductance under the conditions of use.

In a similar manner the whole network can be considered to be divided into self-contained units, the properties of which can, so far as the rest of the network is concerned, be specified in terms of their terminal currents and voltages. Consider the two mutual inductors. Each has two primary and two secondary terminals. Owing to the existence of capacity and eddy currents, the current in the primary winding varies from point to point, and the secondary voltage on open circuit is not exactly equal to the e.m.f. induced by the magnetic field. We may, however, represent the relation between the primary terminal current, I_p, and the secondary terminal voltage on open circuit, V_s, at the frequency $\omega/2\pi$

$$V_s = I_p\,[\sigma + jM\omega] \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3.6)$$

and thus, so far as the circuit external to the instrument is concerned, its behaviour at the frequency $\omega/2\pi$ may be accurately characterized by M, the effective mutual inductance, and σ, the impurity.

The effective values of the self inductance of the resistance coils r and S have already been considered in relation to their terminal currents and voltages, and it will now be understood that the general plan was to construct the apparatus so that the departures from the ideal conditions were as small as possible, and then to measure effective values of the quantities specified in the network, such quantities being defined in terms of the currents and voltages at various definite terminal points of the network. The original equations may then be used with very little modification. Two new terms, σ_1 and σ_2, representing the impurities of M_1 and M_2, must be introduced, in accordance with equation 3.6. We obtain as the working equations

$$\omega^2 [M_1(M_2 + \lambda) + L(l + m)] = Rr + \sigma_1 (S + \sigma_2) \quad . \quad . \quad . \quad (3.7)$$

$$M_1 (S + \sigma_2) + \sigma_1 (M_2 + \lambda) + R (l + m) + rL = 0, \quad . \quad . \quad . \quad (3.8)$$

where all the quantities are to be regarded as effective values, defined as above in terms of the terminal currents and voltages.

But even in establishing this idea of effective values, it has been necessary to introduce the further postulate that every component is infinitely remote from all other objects. When the components are assembled in the actual network, their electric fields are linked and extend as far as the nearest earth-connected body, and the fluctuations of these fields constitute displacement currents flowing from one component to another and from every component to earth. The existence of such currents necessarily complicates our conception of effective values as defined in terms of terminal currents and voltages, since the current entering by one terminal of a bridge component may not all reappear at the other terminal.

We have already seen how such capacity currents internal to each component may affect its properties. We may now distinguish two further capacity effects in the light of the foregoing paragraph. The first can be regarded as internal to the *circuit* as a whole and dependent only on the relative position of the components. By maintaining these relative positions fixed, this effect need never alter and it can be dealt with, as will be shown analytically below, by slightly modifying the effective values of the components. The second effect depends on the position of the components relative to earth. It will be clear that the absolute potentials of the various points of the network, in so far as they determine the flow of current from these points to earth, now become relevant factors in the problem. In practice, it is not desirable, nor yet always possible, to deal with the difficulties arising from this by modifying the equations of balance, but they are overcome by the experimental device described below which produces conditions under which ordinary circuit theory can be applied to the network without modification, and under which the effective values of the bridge components are independent of variations in earth capacities.

The condition of the network assumed in obtaining the equation of balance is that no current flows from the terminals to which the detector is connected except through the bridge components, and it will be evident that this condition can only be realized when there is no current in the detector and also no current flowing to earth from these terminals. In practice this means that the two terminals must be brought to earth potential. This condition is obtained by the use of the Wagner earth connexion (Wagner, 1911), essentially a device for fixing the potential of a bridge network with respect to earth in such a manner that the detector terminals are brought to earth potential, thus satisfying the above conditions. It will be shown analytically and experimentally below that capacity currents from other terminals in the network to earth have no effect on the conditions of balance.

Actually, if by suitable construction of the components and shielding, the whole of the earth-capacity currents of the network may be concentrated at the detector terminals, then the balance conditions are quite unaffected by the earth capacity of the apparatus. It is merely necessary to note that the effective values of the various components must be defined in terms of the current flowing at their terminal which is at earth potential and which therefore carries no earth-capacity current.

There remain two other points for consideration.

We have already seen the importance of the mutual inductance m, between coils P and Q (fig. 1*a*). In practice the best way of dealing with this quantity is to reduce it to zero. The condition to be established is that a current in mesh (1) shall generate no p.d. in mesh (3) except V_r (fig. 1*a*). Thus, if mesh (2) is opened so that i_2 is zero, and V_r is eliminated from mesh (3) by connecting the potential leads of r to the point E, then no p.d. should arise in mesh (3) and G should indicate no current whatever the value of i_1. If the coupling between circuits 1 and 3 in this condition were due only to the mutual inductance, m, between coils P and Q, then the required condition could obviously be easily realized by positional adjustments on these coils. Actually, there exists capacity between the circuits and in addition the unavoidable "massive" metal parts of the assembly (terminals, links, switches, etc.) present, in effect, small closed circuits inductively coupled to both circuits 1 and 2, so that, under the conditions described above, the e.m.f. generated in circuit 3 is not in perfect quadrature with the current in circuit 1. The condition of zero mutual inductance is therefore not sufficient to reduce to zero the unwanted p.d. and a second adjustment is necessary. It is, theoretically, merely necessary to introduce into circuit 3 a small voltage in phase with, or differing in phase by 180° from, the current in circuit 1. This suggests a simple resistance coupling, but obviously, since such a coupling is already present in the complete circuit (r, fig. 1*a*), this is impracticable. The necessary second adjustment is therefore obtained by coupling to circuit 1 a copper ring in which current is induced. This ring is then used as a small potentiometer for the purpose of introducing into circuit 3 a variable p.d. which possesses a component in phase with the current in circuit 1. This "zero adjuster" is described in detail in a later section, and it is shown analytically below that its presence need have no other effect on the working of the apparatus provided that it remains unaltered throughout all the measurements.

(*c*) *Effect of Harmonics*

Finally, we come to a curious practical point arising from the fact that the balance of the bridge is dependent on the frequency of the supply current and that, in order to obtain the required sensitivity of balance, it is necessary to use a thermionic amplifier. If the supply current is not perfectly sinusoidal, then there will be other

" harmonic " frequencies present and the bridge circuit will act as a species of filter for the one frequency, ω, which we will call the " fundamental ", satisfying the equation $\omega^2 M_1 M_2 = Rr$, and currents at the harmonic frequencies $n\omega$ ($n = 2, 3$, etc.) will pass practically without attenuation into the detector circuit. In the first place this gives rise to a complex response of the detecting instrument (in our case a vibration galvanometer), which makes the true balance position extremely difficult to detect. In the second place, the thermionic amplifier may, in virtue of its non-linear characteristic, produce from the harmonic currents a current of the fundamental frequency. To annul this, it is necessary to produce a corresponding current from the bridge by appropriate alterations in the variables, so that a false balance point results. Fortunately, both these effects are immediately recognizable and remediable, for in the first case the complex response is unmistakably obvious and in the second case, although an apparently perfect balance may be obtained, it will be destroyed by reversing the phase of the supply current and a new balance point will be found which may differ considerably from the original. It is, however, not sufficient to regard the mean of these two balance points as being the true one, as might be expected, but the whole of these effects are completely dealt with by the use of suitably designed filter circuits which attenuate the harmonic frequencies to negligible proportions. The theory of this effect is given in the following section.

(*d*) *The Signs in the Equations*

Since the e.m.f. introduced into a circuit via the secondary coil of a mutual inductance may be reversed by a reversal of the direction of winding of the coil, or by reversing the connexions to its terminals, we must regard M in the equations for any network as capable of taking either a positive or negative sign. Similarly the p.d. introduced into a circuit by the potential terminals of resistances such as r and S may be reversed by a reversal of the connexions to these terminals, and therefore r and S (and also the corresponding residual inductances l and λ) may be regarded as taking either a positive or negative sign in the equations which we have given for the Campbell network. In the determination of resistance by means of the basic equation $Rr = M_1 M_2 \omega^2$ no ambiguity arises on account of the question of sign, but in dealing with the complete equations 3.2 and 3.3, attention must be paid to the point. In deriving our equations the following conventions were adopted. Let P_1 and P_2 denote the primary terminals, and S_1 S_2 the secondary terminals of the mutual inductor M_1, and let P_1 and S_1 be the terminals which are connected together. If the current flowing from P_1 to P_2 is I_{12} and V_1 and V_2 are the potentials of S_1 and S_2, the working equation of the instrument should be written

$$V_{12} = V_1 - V_2 = (\sigma + jM\omega)\, I_{12} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3.9)$$

which defines the sign of M and σ. This convention implies that a mutual inductance is positive when the direction of winding is such that, one pair of terminals being common, if current is passed through the two coils via the open ends, their mutual inductance diminishes their total self inductance.

The quantities r, l, S, λ were all regarded as positive, when their connexions in the network were those shown in fig. 1, and their working equations were, in a notation exactly similar to the above,

$$V_{12} = V_1 - V_2 = (r + jl\omega)\, I_{12} \quad . \; . \; . \; . \; . \; . \; . \; . \quad (3.10)$$

It will now be apparent from (3.2) and (3.3) that the network can only be balanced when certain combinations of the signs of M_1, M_2, r, and S obtain. Thus, if we make r positive, M_1 and M_2 must both be of the same sign by (3.2). If S is also positive, then M_1 must be negative by (3.3), and therefore both M_1 and M_2 must be negative. Other possible combinations of sign may readily be obtained by reversing the connexions to the potential terminals of r and S in turn. A Table is drawn up below showing the four possible combinations of signs which may be used. Several of the small correcting terms may be ignored in this connexion—m, because in practice it was reduced to zero, $\sigma_1\sigma_2$, because it is a small quantity of the second order of magnitude, and the small terms in equation (3.3), because they are not required for the calculation of the result. It should be noted that the signs of l and λ are necessarily changed with those of r and S respectively.

Table I—The Possible Signs of the Quantities in the Equations of Balance

$$\omega^2\,[M_1\,(M_2 + \lambda) + Ll] = Rr + S\sigma_1 \qquad M_1S + Lr = 0$$

	M_1	M_2	λ	Ll	Rr	$S\sigma_1$	M_1S	Lr
a	+	+	−	++	++	−+	+−	++
b	−	−	+	++	++	++	−+	++
c	+	−	+	+−	+−	++	++	+−
d	−	+	−	+−	+−	−+	−−	+−

Since the value of σ_1 can be determined only by experiment, it has been given an arbitrary positive sign in the above table. It will be seen that the possible arrangements fall into two groups, one in which M_1 and M_2 have the same sign and r is positive, and one in which M_1 and M_2 have opposite signs and r is negative. The circuit arrangement for the first group is that given in fig 1*a* and is Campbell's original scheme. That for the second group is given in fig. 2. It is clear from the fig. 1*a* that the only way of bringing the detector circuit to earth potential for this arrangement is by directly connecting the point E to earth. This must be rejected as it does not conform to the conditions previously laid down for making the operation of the network independent of earth-capacities. The second arrangement, however, permits the use of the Wagner earthing arm previously referred to and allows the point F (fig. 2) to be brought to earth potential without being directly earthed. The Wagner arm consists of two impedance components connected in series across the supply leads with their common point earthed. These components, together with the coil P and resistor r, form an ordinary four-arm bridge which may be balanced with a detector between F and earth. The actual components are shown in fig. 2. The arm is made as far as possible non-inductive, so that the arrangement shown was preferred to the more obvious one in which the Wagner arm would have been made up of an inductance and resistance corresponding to P and r respectively. Since there are four inductive coils in the network, there arise eight possible methods of connexion which will give quantities with signs corresponding to the arrangements c and d of Table I in which r is negative, for it is evident that the sign of the mutual inductance of any two coils is unaltered if the connexions to both coils are reversed. Thus, if we denote the four coils of the

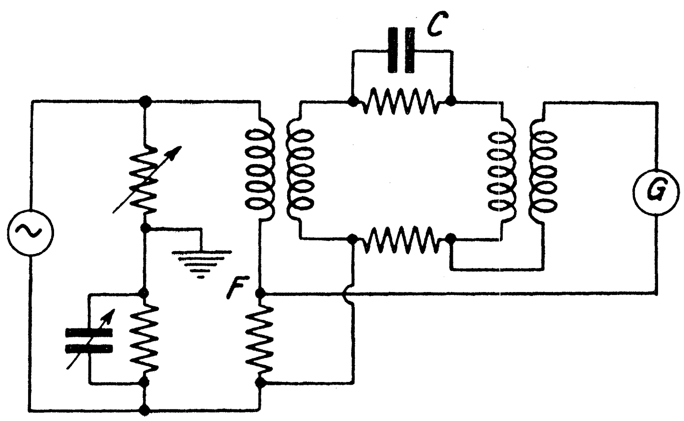

FIG. 2—Campbell circuit with Wagner earthing arm.

network by P, N, O, and Q (*see* fig. 3), and indicate a reversal of the connexions to any one coil by a change from \ to /, we have for the eight arrangements Table I (a).

TABLE I (a)

Arrangement		P	N	M_1	O	Q	M_2
c	1	\	\	+	\	\	−
	2	/	/	+	\	\	−
	3	/	/	+	/	/	−
	4	\	\	+	/	/	−
d	5	\	/	−	\	/	+
	6	/	\	−	\	/	+
	7	/	\	−	/	\	+
	8	\	/	−	/	\	+

If all the corrections have been successfully estimated and applied, there should be no differences between the results for the eight connexions. Actually the differences obtained were greater than could be accounted for by purely observational errors, but it was found that, although the individual values were changed by alterations in the wiring such as would cause slight changes in the stray mutual inductances in the network, the mean of the eight values remained constant within the observational error. It is therefore considered that the eight sets of observations

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should be regarded as a single complete determination, this procedure being necessary in order to eliminate small uncertainties which could not be reduced to zero or measured separately.

The final working equation expressed numerically on the basis of the signs given above is

$$\omega^2 [M_1 (M_2 - \lambda) + Ll] = (R + \Delta R)\, r + S\sigma_1 \quad . \quad . \quad . \quad . \quad (3.11)$$

in which Δ R has the significance given to it in equation (3.5).

This is conveniently written for computation in the form

$$M_1 M_2 (1 + \psi) = Rr/\omega^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3.12)$$

in which

$$\psi = \frac{1}{M_1 M_2}\left[(Ll - M_1 \lambda) - \frac{r\Delta R \pm S\sigma_1}{\omega^2}\right], \quad . \quad . \quad . \quad . \quad . \quad . \quad (3.13)$$

and represents the correction to be applied to the basic equation (2.1).

The practical limitations of the Campbell method, which are common to all alternating current bridge methods, may perhaps suggest that such methods are very complicated and uncertain. However, the effects discussed are all very small and in the present investigation the largest term in the correcting factor amounted to $3{\cdot}4 \times 10^{-5}$ and the algebraic sum of all the terms to about 3×10^{-5}. The final accuracy hoped for is 1 part in 100,000, so that even if ψ is known to not better than 10% or 20%, the final result will not be greatly affected.

4—Analysis of the Circuit

We have seen that departures from the ideal network due to eddy currents, displacement currents, and stray inductance are to be dealt with by replacing the simple inductances and resistances of the ideal circuit by impedance operators, each referring to some portion of the actual network between specified points. The zero adjuster already mentioned is essentially a device for adding an eddy-current effect of controllable magnitude to mesh (2). Its presence will modify the impedance operators of the three meshes, and, provided these impedance operators are measured with the adjuster set in its proper position, the fact that it constitutes a fourth mesh may be ignored in practice. This procedure is justified by the following analysis of the network.

Fig. 3 represents the Campbell circuit including the zero-adjusting circuit, indicated as mesh (4), and the various associated capacities.

Let ξ_2, ξ_3, ξ_4 be the impedance operators for circuits 2, 3, and 4.

Let μ_{12} be the mutual impedance operator for circuits 1 and 2, and let all μ's have similar meanings, except that μ_{23} does not include the part due to the component S, which is called ξ_{23}, and μ_{13} does not include the part due to the component r, which is called ξ_{13}. Ignoring for the present all the capacitances to earth, we may proceed as follows.

For the actual zero adjustment the lead p_2 is transferred to F (fig. 3) and mesh (2) is opened. Then for balance

$$\mu_{13} i_1 + \mu_{34} i_4 = 0, \quad (4.1)$$

$$\mu_{14} i_1 + \xi_4 i_4 = 0, \quad (4.2)$$

whence

$$\mu_{13} = \mu_{14}\, \mu_{34}/\xi_4. \quad (4.3)$$

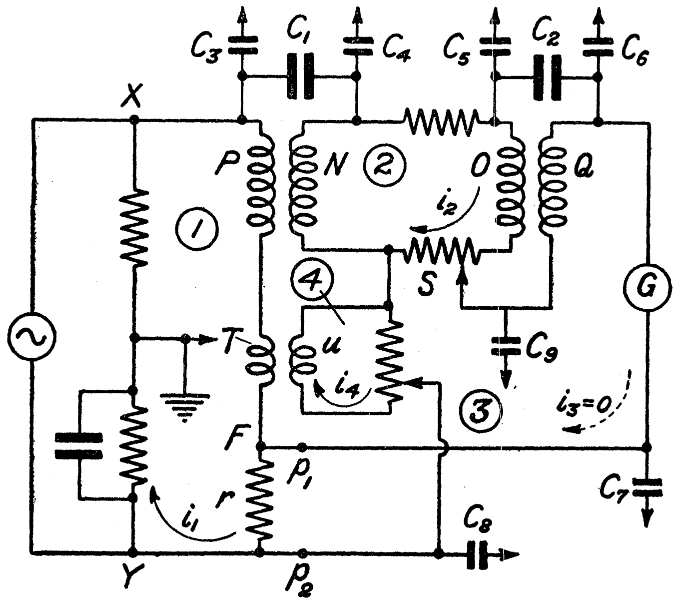

Fig. 3—Campbell circuit with zero-adjusting mesh, earth capacitances and intercapacitances.

For the main bridge balance, p_2 is restored to its original position, mesh (2) closed, and the zero adjustments left unaltered. Then for balance

$$\xi_2 i_2 + \mu_{24} i_4 + \mu_{12} i_1 = 0, \quad (4.4)$$

$$\mu_{24}\, i_2 + \xi_4 i_4 + \mu_{14} i_1 = 0, \quad (4.5)$$

$$(-\mu_{23} + \xi_{23})\, i_2 + \mu_{34} i_4 + (-\mu_{13} + \xi_{13})\, i_1 = 0. \quad (4.6)$$

Eliminating i_1, i_2, and i_4 from these equations, we get

$$(-\mu_{23} + \xi_{23})\,(\mu_{24}\mu_{14} - \xi_4\mu_{12}) + \mu_{34}\,(\mu_{24}\,\mu_{12} - \xi_2\,\mu_{14}) + (\xi_2\xi_4 - \mu_{24}^2)\,(-\mu_{13} + \xi_{13}) = 0. \quad (4.7)$$

Substituting for μ_{13} from (4.3), we get, collecting and rearranging terms,

$$-\xi_{13}\left(\xi_2 - \frac{\mu_{24}^2}{\xi_4}\right) + \left(\mu_{12} - \frac{\mu_{14}\mu_{24}}{\xi_4}\right)\left(-\mu_{23} - \xi_{23} + \frac{\mu_{24}\mu_{34}}{\xi_4}\right) = 0. \quad (4.8)$$

Now the main mutual impedances μ_{12} and μ_{23} are determined in the following way. The second, μ_{23} is balanced directly against the primary standard, μ_s, the original circuits being retained and arranged as in fig. 4*a*. For balance, we have

$$(\xi'_{23} - \mu_{23})\, i_2 + \mu_s i_2 + \mu_{34} i_4 = 0, \quad (4.9)$$

$$\xi_4 i_4 + \mu_{24} i_2 = 0, \quad (4.10)$$

whence
$$\left(\xi'_{23} - \mu_{23} - \frac{\mu_{24}\mu_{34}}{\xi_4}\right) + \mu_s = 0. \quad (4.11)$$

For this balance the potentiometer S takes up a new setting. We have therefore changed ξ_{23} to ξ'_{23}.

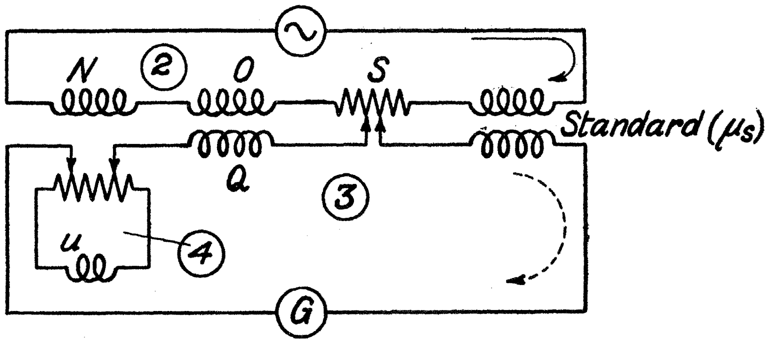

FIG. 4—(*a*) Measurement of μ_{23}.

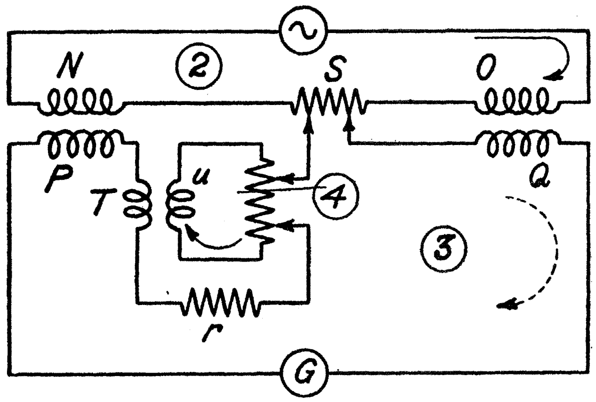

FIG. 4.—(*b*) Measurement of μ_{12}.

Similarly μ_{12} is obtained in terms of μ_{23}, the circuits being arranged as in fig. 4*b*. For this, we have, if δ represents the small amount by which μ_{23} must be altered to obtain balance, and if ξ''_{23} corresponds to the new setting of S

$$\{-(\mu_{23} + \delta) + \xi''_{23}\}\, i_2 + \mu_{12} i_2 + (\mu_{14} + \mu_{34})\, i_4 = 0, \quad (4.12)$$

$$\xi_4 i_4 + \mu_{24} i_2 = 0, \quad (4.13)$$

whence

$$-\mu_{23} + \xi''_{23} + \mu_{12} - \delta - \mu_{24}(\mu_{14} + \mu_{34})/\xi_4 = 0. \quad (4.14)$$

Substituting from (4.11), we find

$$\left(\mu_{12} - \frac{\mu_{14}\mu_{24}}{\xi_4}\right) - \mu_s - \delta + \xi''_{23} - \xi'_{23} = 0. \quad (4.15)$$

Equation (4.8) may now be re-written in terms of the measured quantities μ_s and δ

$$-\xi_{13}\xi'_2 + (\mu_s + \delta + \xi'_{23} - \xi''_{23})(\mu + \xi'_{23} - \xi_{23}) = 0, \quad (4.16)$$

where $\xi'_2 = \xi_2 - \mu_{24}{}^2/\xi_4$. Thus ξ'_2 is the impedance operator of mesh (2) with mesh (4) in position. If R_e and L_e are the effective resistance and inductance of mesh (2) measured with mesh (4) in position, we have

$$\xi'_2 = R_e + j\omega L_e. \quad (4.17)$$

Also

$$\xi_{13} = -(r + jl\omega).^* \quad (4.18)$$

$$\mu_s = \sigma_2 + j\omega M_2 \quad (4.19)$$

where M_2 and σ_2 are the effective values of mutual inductance and impurity measured as described above.

Similarly,

$$\mu_s + \delta = -(\sigma_1 + j\omega M_1)^* \quad (4.20)$$

The settings of S corresponding to ξ'_{23} and ξ''_{23} are in practice almost identical and we may therefore put

$$\xi'_{23} - \xi''_{23} = 0. \quad (4.21)$$

On the other hand the settings corresponding to ξ_{23} and ξ'_{23} are very different and we may write

$$\xi_{23} - \xi'_{23} = -(S + j\omega\lambda)^*. \quad (4.22)$$

where S is the difference between the two resistance-readings of the potentiometer (*a*) at the setting required for the main bridge balance and (*b*) at that required for the two auxiliary balances (practically the zero reading); and λ is the difference between the values of the residual inductance of this instrument corresponding to the same two settings.

Substituting these values in equation (4.16), we obtain the working equation

$$(r + j\omega l)(R_e + j\omega L_e) - (\sigma_1 + j\omega M_1)(\sigma_2 + j\omega M_2 + S + j\omega\lambda) = 0, \quad (4.23)$$

leading to

$$rR_e - Ll\omega^2 + M_1M_2\omega^2 - \sigma_1(S + \sigma_2) + M_1\lambda\omega^2 = 0. \quad (4.24)$$

This equation should be compared with that given for the simpler network (equation (3.7)). Apart from the signs, the conventions with regard to which have already been discussed, the equations are identical. We may therefore ignore the complications introduced by the zero adjuster provided that the various quantities are measured in the way described.

We have now to consider the effect of mutual capacitances between the various components of the network and capacitances between the various components and earth.

* The signs adopted here correspond to one of the arrangements actually used and given at (*d*), Table I.

Since we have referred all our impedances, etc., to the current values at certain points of the network, we may represent the distributed capacitance of the whole system by lumped capacitances concentrated at these points. Thus the whole internal capacitance, apart from capacitance to earth, may be represented by C_1 and C_2 in fig. 3. One effect of these is, as has already been pointed out, to modify the values of M_1 and M_2. Another effect, as shown by BUTTERWORTH (1921), is to augment the resistance of the secondary winding. Both these effects are fully taken into account by the measurements of the variation of mutual inductance and resistance with frequency, as described in a later section of the paper.

We may also represent the effect of capacitances to earth by the lumped capacitances C_3 to C_9 shown in fig. 3. Of these C_3 and C_8 merely form part of the earthing arm, XY, and so are outside the bridge, while C_6 and C_7 carry no current, since the points to which they are connected are at earth potential when balance is established. They may therefore be ignored. C_4 and C_5 may, by a simple delta-star transformation, be replaced by a single capacity, the impedance of circuit 2 being slightly modified. This equivalent capacity, together with C_9, represents then the chief effects of earth capacitance.

A rough estimate of the errors arising from C_9 may be obtained as follows. Its effect will be to alter the current through S, which will no longer be i_2 but some quantity $i_2 - \varepsilon$, where ε is given by

$$\begin{aligned}\varepsilon &= (ri_1 + \mathrm{S}i_2)\, j\omega \mathrm{C}_9 \\ &= i_2 j\omega \mathrm{C}_9\left(\frac{r}{j\omega \mathrm{M}_1}\ (\mathrm{R} + j\omega \mathrm{L}) + \mathrm{S}\right) \\ &= i_2\,\frac{\mathrm{R}r\mathrm{C}_9}{\mathrm{M}_1}, \qquad (4.25)\end{aligned}$$

since we may write $\mathrm{L}r + \mathrm{M}_1\mathrm{S} = 0$ (equation (2.2)).

The effective value of S is therefore altered by the fraction $\mathrm{R}r\mathrm{C}_9/\mathrm{M}_1$, which is only of the order of a few parts in 10^6 if C_9 is as high as 1000 $\mu\mu$F. In any case, the main conditions of balance are not affected, since the value of S does not occur except in very small correcting terms. In the same way the capacitance to earth from the other side of the network can be shown to exert a negligible influence on the balance conditions. These deductions are substantiated by experimental tests in which it was shown that an added capacitance to earth of as high as 10,000 $\mu\mu$F was required to make the smallest detectable change in the balance point, and the existence of such a large capacitance as this is manifestly improbable.

It only remains to consider analytically the effect of harmonics briefly mentioned in the preceding section, *i.e.*, the effect on the balance point of a supply current that is not perfectly sinusoidal and an amplifier of non-linear characteristic placed between the bridge and the detector. The voltage impressed on the amplifier by the bridge will be of the form

$$v = v_1 \sin(\omega t + \alpha) + v_2 \sin(2\omega t + \beta) + v_3 \sin(3\omega t + \gamma) + \ldots \qquad (4.26)$$

where the v's represent voltage amplitudes and α, β, γ are arbitrary phase angles. The current, i, produced by the amplifier and passed into the detector, will be related to v by a relationship of the form

$$i = av + bv^2 + cv^3, \quad \ldots \ldots \ldots \quad (4.27)$$

where a, b, c are constants characteristic of the amplifier, and in practice the first term is large compared with all the others.

Substituting from (4.26) and (4.27) and rearranging terms, we find that the current, i_ω, of frequency ω (*i.e.*, the fundamental frequency) passing into the detector is given by, as far as second order terms,

$$i_\omega = \{av_1 \sin(\omega t + \alpha) + 2bv_1v_2 \cos(\omega t + \beta - \alpha) + 2b\, v_2v_3 \cos(\omega t + \gamma - \beta) \ldots$$

which may be written in the usual vector notation as

$$i_\omega = \{av_1 \sin\alpha + 2bv_1v_2 \cos(\beta - \alpha) - 2bv_2v_3 \cos(\gamma - \beta)\} + j\{av_1 \cos\alpha + 2bv_1v_2 \sin(\beta - \alpha) - 2bv_2v_3 \sin(\gamma - \beta)\}. \quad (4.28)$$

When the detector is balanced, by adjustments on the bridge, then $i_\omega = 0$, and we must then have

$$\left. \begin{aligned} av_1 \sin\alpha + 2bv_1v_2 \cos(\beta - \alpha) - 2bv_2v_3 \cos(\gamma - \beta) = 0 \\ av_1 \cos\alpha + 2bv_1v_2 \sin(\beta - \alpha) - 2bv_2v_3 \sin(\gamma - \beta) = 0 \end{aligned} \right\} \quad . \; . \quad (4.29)$$

These two equations are sufficient to determine the magnitude and phase of the voltage of fundamental frequency (v_1) which must be impressed on the amplifier to produce balance. Solving, we get, vectorially

$$v_1 = \frac{2bv_2v_3}{a^2 - 4b^2v_2^2} \{[a \sin(\gamma - \beta) - 2bv_2 \cos\gamma] - j[a \cos(\gamma - \beta) - 2bv_2 \sin\gamma]\}, \quad \ldots \quad (4.30)$$

Thus, if there are no harmonics ($v_2 = 0 = v_3$) or the amplifier has a linear characteristic ($b = 0$), then v_1 will be zero, as would be expected.

Reversing the connexions to bridge or amplifier alters the phase of the impressed voltage by π and in order to obtain balance, a new value of v_1, say v_1', must be produced. This value is simply obtained by writing $-a$ for a in equation (4.30). This gives

$$v_1' = \frac{2bv_2v_3}{a^2 - 4b^2v_2^2} \{[-a \sin(\gamma - \beta) - 2bv_2 \cos\gamma] + j[a \cos(\gamma - \beta) + 2bv_2 \sin\gamma]\} \quad \ldots \quad (4.31)$$

The difference between v_1 and v_1' is

$$\delta v_1 = \frac{4abv_2v_3}{a^2 - 4b^2v^2} \{\sin(\gamma - \beta) - j \cos(\gamma - \beta)\}, \quad \ldots \quad (4.32)$$

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and the mean value is

$$\bar{v}_1 = \frac{-4b^2v_2{}^2v_3}{a^2-4b^2v_2{}^2}\{\cos\gamma + j\sin\gamma\}. \quad (4.33)$$

Now δv_1 will be produced by displacements δM_2 and δS of M_2 and S, the bridge variables, such that

$$\delta v_1 = \eta_m \delta M_2 + j\eta_s \delta_s \quad (4.34)$$

where η_m and η_s are respectively the voltage sensitivities to the fundamental frequency of the network to changes in M_2 and S.

Thus the differences between the two readings of M_2 and of S before and after reversal will be given by

$$\delta M_2 = \frac{1}{\eta_m}\frac{4bv_2v_3}{a_2-4b^2v_2{}^2}\sin(\beta-\gamma) \doteqdot \frac{1}{\eta_m}\frac{4b}{a}v_2v_3\sin(\beta-\gamma), \quad (4.35)$$

$$\delta S = \frac{1}{\eta_s}\frac{4bv_2v_3}{a^2-4b^2v_2{}^2}\cos(\beta-\gamma) \doteqdot \frac{1}{\eta_s}\frac{4b}{a}v_2v_3\cos(\beta-\gamma), \quad (4.36)$$

the approximate forms arising from the assumption that $b^2v_2{}^2 \ll a^2$ in general. The mean readings of M_2 and of S will differ from the readings corresponding to the true balance point by quantities ΔM_2 and ΔS such that

$$\Delta M_2 = \frac{1}{\eta_m}\frac{4b^2v_2{}^2v_3}{a^2-4b^2v_2{}^2}\cos\gamma, \quad (4.37)$$

$$\Delta S = \frac{1}{\eta_s}\frac{4b^2v_2{}^2v_3}{a^2-4b^2v_2{}^2}\sin\gamma, \quad (4.38)$$

and provided that $\beta-\gamma$ is neither 0 nor $\pi/2$, we may write

$$\Delta M_2 = (b/a)\,v_2\,\delta M_2\cos\gamma/\sin(\beta-\gamma), \quad (4.39)$$

$$\Delta S = (b/a)\,v_2\,\delta S\sin\gamma/\cos(\beta-\gamma). \quad (4.40)$$

In practice, readings were always taken before and after reversal of the voltage applied to the bridge, the differences δM_2 and δS were noted, and the mean readings were taken as correct. It follows from equations (4.30) that the fractional errors of these mean readings due to harmonics are given by

$$\frac{\Delta M_2}{M_2} = \frac{v_2}{v}\frac{bv^2}{av}\frac{\delta M_2}{M_2}\frac{\cos\gamma}{\sin(\beta-\gamma)}, \quad (4.41)$$

and a similar expression for $\Delta S/S$. Now v_2/v depends on the harmonic content of the applied voltage. It is reduced to a very small value by the filter circuit and was found to be of the order of 0·01. The term bv^2/av is a fraction expressing the deviation of the characteristic of the amplifier from a straight line. By the choice of suitable valves and bias voltages, this also can be reduced to a few per cent. Thus the product of the first two terms on the right-hand side of (4.41) amounts to only a few parts in 10^4. With regard to the remaining terms, we see from (4.35) that δM_2 for a given harmonic content is proportional to $\sin(\beta - \gamma)$ and therefore varies with the relative phases of the harmonics. It is a simple matter to vary the conditions in the oscillator amplifier or filter circuits, and thus to obtain a few values of $\delta M_2/M_2$ corresponding to various typical values of $\cos\gamma/\sin(\beta - \gamma)$. In this way the order of $\delta M_2 \times \cos\gamma/M_2 \sin(\beta - \gamma)$ was shown to be less than 1×10^{-4}, so that the error due to harmonics is likely to be only a few parts in 10^8, with the circuits actually employed.

5—Instrumental Details

The equipment necessary for the investigation may conveniently be considered under three heads :

(i) The primary standards of mutual inductance and frequency.
(ii) The bridge components.
(iii) The source of current and detector of balance.

The Campbell primary standard of mutual inductance has been completely described elsewhere (Campbell, 1907, 1912 ; Dye and Hartshorn, 1927) and no further detail is necessary here. The primary standard of frequency used for this investigation is an electrically-maintained tuning-fork, vibrating at a frequency of 1000 cycles per second, continuously checked against astronomical time signals. This also has been completely described elsewhere. (Dye and Essen, 1934.)

The bridge components required are two mutual inductances, one of which may be fixed, the other being capable of continuous variation over a small range, a standard four-terminal resistance (r), a low resistance potentiometer (S), and a zero-adjusting circuit.

(*a*) *Magnitude of Bridge Components*

The magnitude of the main components is fixed by the following considerations. The primary standard of mutual inductance has a value of 10 millihenries, so that this should also be the value of the mutual inductances in the bridge, in order that their values may be determined in terms of the standard with the highest possible precision. The choice of the working frequency is in the nature of a compromise, for the frequency must be sufficiently low to make the corrections due to capacity and eddy currents very small and yet high enough to permit the use of a detector of high sensitivity and quick response. These requirements are most conveniently met by a frequency of 100 cycles per second, and at this frequency, with mutual inductances of 10 mH each, it appears from equation (2.1) that the product Rr is approximately 40. For precision and convenience in general standardizing work it is desirable that one of the resistances, R, r, shall have the value 1 ohm, but since R includes the resistances of two of the coils in the inductances and also the potentio-

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meter S, it is obvious that r should be chosen to have this value, and that the value of R should therefore be about 40 ohms. Since R must be measured with great precision, it is important that its temperature coefficient should be very small. The use of manganin wire for the whole of circuit (2), including the appropriate coils of M_1 and M_2, was considered, but was found to limit too severely the possible values of inductance. It was therefore decided to wind the inductive coils of copper wire of as low a resistance as possible, and to place in series with them manganin resistance-coils adjusted to give the required total value of resistance.

A frequency of 50 cycles per second was also used, but although the value of R for this frequency was 10 ohms, and therefore very convenient for comparison with the Laboratory standards, the arrangement was in practice less satisfactory, for the proportion of copper in R was increased, making the value correspondingly less stable. Also the alternating current measurements were made tedious and difficult because of the interference from neighbouring supply mains carrying current at 50 cycles per second.

The combined inductance of the secondary mesh was approximately 10 mH, so that, substituting this value for L in equation (2.2), we find that the value of S should be approximately 1 ohm.

(*b*) *The Mutual Inductances* M_1 *and* M_2

These two instruments were identical in general design, the only difference between them being that to the secondary of M_2 was added a small coil mounted on a rotating arm in such a manner that the rotation caused the mutual inductance between it and the primary coil to vary continuously over the range $\pm$ 5 microhenries. A pointer rigidly attached to the moving coil indicated the value of the added mutual inductance on a scale which could be read to 0·01 microhenry, *i.e.*, to 1 part in a million of the value of M_2. This moving coil provided the fine adjustment of M_2 required for balancing the network.

The form of the main coils was determined by the following considerations: one winding (that to which we have already referred as the secondary) must possess a very low resistance, and the effects of eddy currents and capacity currents on this resistance, and on the value of the mutual inductance and its impurity, must be kept small. Eddy currents were made small by the use of stranded copper wire, 81 strands of No. 36 S.W.G. (0·2 mm. diam.) (each strand having single enamel insulation with double silk insulation over all), for the low resistance coils, and 27 strands of No. 36 for the primary coils. Capacity effects were minimized by keeping the self inductance of each coil as low as possible. This condition prevented us from using only a few turns in the secondary winding and thereby making its resistance extremely small, since a reduction in the number of turns of the one winding would have necessitated an increase in that of the other. The coupling coefficient of the two coils should evidently be large, since this condition also makes for coils of small self inductance; the primary coils were therefore wound in two halves placed one on either side of the secondary coil. The bobbins are built up from plates and disks of marble, cemented together with bakelite varnish. After winding, the coils were heated for some hours in transformer oil at a temperature of 110°–120° C. In this way they were thoroughly dried, and strains in the windings were as far as possible removed. A standard so treated and afterwards kept in transformer oil shows remarkable stability and freedom from defects of insulation. While the change during the heating may amount to several parts in a thousand, the subsequent value remains stable to a few parts in a hundred thousand for a year or more.

The secondary winding, on the middle section of the bobbin, possesses about 150 turns, has an inductance of 5 mH, and a resistance of 0·4 ohms. The primary winding occupies the outer sections and has an inductance of about 40 mH, and a resistance of about 10 ohms. A small coil, on a keramot* former bolted to one cheek of the bobbin and coaxial with the main windings, is included in the primary winding to permit the final adjustment of inductance being made with the desired precision. The values of mutual inductance were adjusted to be within 2 parts in 10,000 of 10 mH.

(*c*) *The Resistor r*

This four-terminal resistor should possess at once the stability of resistance of a standard for direct-current work and the freedom from inductance and eddy current losses required of a standard for alternating current work. These requirements are very exacting and the most successful of many arrangements is described below.

A manganin coil of resistance 1 ohm, annealed in nitrogen in the manner described by Rayner (1935), was mounted on a keramot holder, instead of the more usual metal one, supported from a keramot panel carrying the current leads and the potential terminals. The former consisted of copper bars of cross-section about 1 cm. square, parallel to each other and about 3 mm. apart. The coil, although wound in a bifilar manner, had an inductance of nearly 0·7 μH. This was, however, largely compensated by the following method. A small toroid, of mean radius 1 cm., was uniformly wound with two windings, one of which was included in a current lead and the other in the corresponding potential lead, the connexions being arranged so that the mutual inductance of the toroidal coils opposed the self inductance of the resistance coil. Short lengths of copper wire were added to the other pair of leads, since it is desirable from the point of view of convenience in measurement that the leads shall have approximately equal resistances.

The residual inductance, as defined by equation (3.10), was $0{\cdot}01_7$ μH, and the stability of resistance was of the order of a few parts in 10^6 per year.

(*d*) *The Potentiometer* S

We have already seen that S must have a value of about 1 ohm. This value need not be known with any accuracy, but the value λ of its effective self inductance

* A hard rubber compound selected as being free from surface deterioration.

must be small and definite. In addition, S must be capable of almost as fine adjustment as M_2. The potentiometer is also used in the circuit arrangements which are used for the measurement of M, and here the value of S required is of the order of $\pm$ 10^{-4} ohm (*see* sections 7 and 9). These features were combined in the following way. The potentiometer was made in two parts, one a fixed resistance of about 1 ohm, wound in a straight bifilar loop and the other a circular slide wire (actually a manganin strip) of 30 cm. diameter, and about 0·07 ohms resistance and readable to a thousandth part of the whole. For using the potentiometer in the main bridge, the potential tappings come from the slide wire and from one end of the fixed resistance (P_2 and P_1 in fig. 5) : for the inductance comparison bridges the fixed potential tapping is taken from a position P_3 roughly halfway along the slide wire.

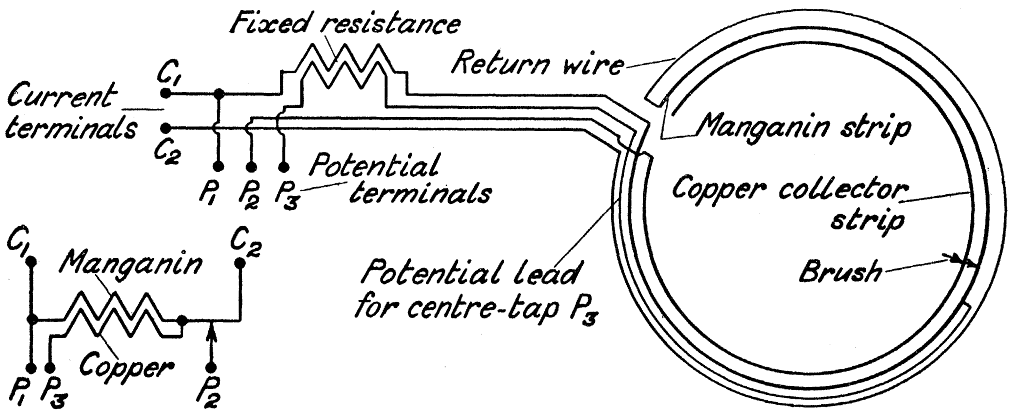

FIG. 5—Diagram of Potentiometer S.

The arrangement is shown diagrammatically in fig. 5, from which it will be seen that the leads are disposed so as to leave no open loop in the system, and therefore to minimize stray mutual inductance, and that the potential leads follow the current circuit as closely as possible, that to P_3 actually duplicating geometrically the fixed resistance part. This is in accordance with CAMPBELL's method (1908) of compensation for non-inductive shunts, the mutual inductance between current and potential leads opposing the self inductance of the current circuit.

As has been shown (equation (4.22), the actual correcting term appearing in the final equations is really the difference between the values of effective inductance, as defined by equation (3.10) for the two arrangements.

(*e*) *The Zero-adjusting Circuit*

The object of the zero adjuster is to introduce into mesh (3) of the circuit an e.m.f. in phase with and proportional to i_1, fig. 1*a*. A transformer with a secondary winding of a resistance that is negligible in comparison with its reactance suggests itself as a possible solution of the problem. After several trials the following arrangement was found to be satisfactory. A few turns of wire included in circuit 1 were arranged so as to induce current into a secondary circuit consisting of a copper ring of mean diameter 10 cm. and of cross-section 1 cm. square. The ring was also provided with fixed and sliding contacts so that it formed a small potentiometer, from which the required variable p.d. could be obtained. The two potential-contacts were included in mesh (3). In order that the p.d. at the potential terminals of this instrument shall be in phase with the current in the ring, it is essential that the mutual inductance between the potential lead and current path shall always compensate the self inductance of the latter in accordance with CAMPBELL's principle, previously mentioned (1908). This condition was approximately satisfied by using for the adjustable potential lead a hollow brass ring almost completely surrounding the copper ring, with about 0·5 mm. clearance all round, as shown in fig. 6. A rotating brush established contact between ring and tube, which was slotted in order to allow the brush to pass through to the ring. The details will be clear from fig. 6. The number of turns required on the primary winding was determined by trial.

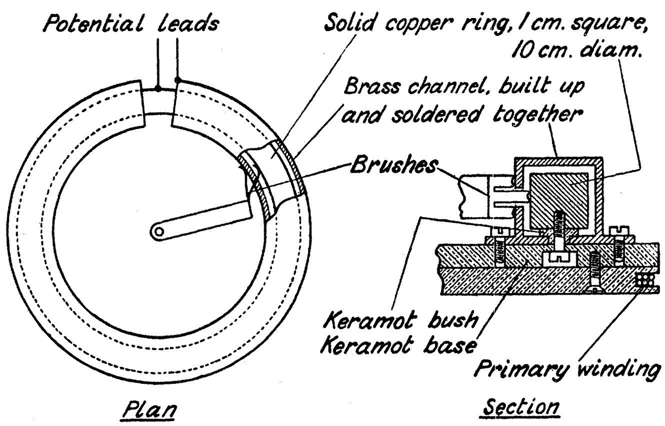

FIG. 6—Diagram of Zero-adjusting Potentiometer.

(*f*) *The Source of Current*

The chief requirements of the source of current are stability of frequency, freedom from harmonics, and adequate available power. The first is conveniently obtained by the use of a tuning-fork and the other two by the use of an alternator, and in our early experiments an attempt was made to combine these in a controlled alternator set, but the results were not sufficiently good for our purpose. A tuning-fork oscillator was therefore developed, the fork being maintained in vibration by the method described by HODGKINSON (1925), and suitable amplifying and filtering circuits added.

The actual oscillator contains three elinvar forks of frequencies 50, 100, and 200 cycles per second, which can be used in turn. The fork is maintained in vibration by one valve, which is followed by a buffer circuit. The output from this is applied to two amplifiers in parallel, one of which is used to supply the small amount of power used in the frequency-measuring gear (*see* section 7), the other supplying the bridge network. The output from this second amplifier passes through a filter circuit which reduces the total harmonic content to the order of 1% of the output. The available power is about 2 watts.

(*g*) *The Detector Circuit*

The detector circuit consists of a highly selective thermionic amplifier followed by a vibration galvanometer of the CAMPBELL type. The instrument used was designed and constructed specially for this investigation. The moving system consists of a rectangular coil about 5 mm. long and 1 mm. wide, of about 15 turns of 0·05 mm. diameter wire, and carrying a small mirror, about 1 mm. square.

The actual amplifying circuit is preceded and followed by low-pass filter circuits, which produce an attenuation of about 20 decibels at 150 cycles per second, and 100 decibels at 200 cycles per second when arranged for a working frequency of 100 cycles per second. The input transformer of the amplifier has a high step-up ratio (200/1) and the secondary winding is brought into resonance with the working frequency by an adjustable condenser connected across it. The output transformer is chosen to match the impedance of the valve to that of the filter circuit and detector.

6—THE ASSEMBLY OF THE COMPONENTS AND DISPOSITION OF THE APPARATUS

The complete network must be provided with arrangements for carrying out the following measurements, without appreciably changing the significant properties of the three component circuits.

(*a*) The balancing of Campbell bridge (fig. 7*a*).

(*b*) The measurement of M_2 by comparison with the primary standard (fig. 7*b*).

(*c*) The measurement of M_1 by comparison with M_2 (fig. 7*c*).

(*d*) The measurement of the resistance R of mesh (2) in terms of resistance standards.

(*e*) The measurement of resistance r in terms of resistance standards.

The last measurement (*e*) is a simple comparison of 1 ohm coils, and is easily done by detaching the resistor r from mesh (1) when necessary. The value of r is of course quite unaffected by this procedure. The other measurements, however, demand conditions which are less easily satisfied. We must, for example, be able to remove the source of current from mesh (1) (fig. 7*a*) and insert it into mesh (2) either at the point g_1 (fig. 7*b*) or at g_3 (fig. 7*c*), and this must be done without appreciably altering the contour of either mesh. Moreover, although we must be able to break circuit (2) in this way at two points at least, its resistance must be stable to 1 part in 10^6 or to 4×10^{-5} ohm, and must be reproduced to this accuracy on making and breaking the circuit, otherwise the measurement (*d*) could not be made with the necessary accuracy. It must also be possible to connect the windings of the primary standard of mutual inductance in series with meshes (2) or (3) without appreciably changing their mutual inductance, and therefore their geometry. These series-additions to the meshes are carried out as follows. A very small gap is made in the circuit in question and the additional component, which is at a considerable distance away, is connected to the gap by a pair of closely twisted leads. The gaps and

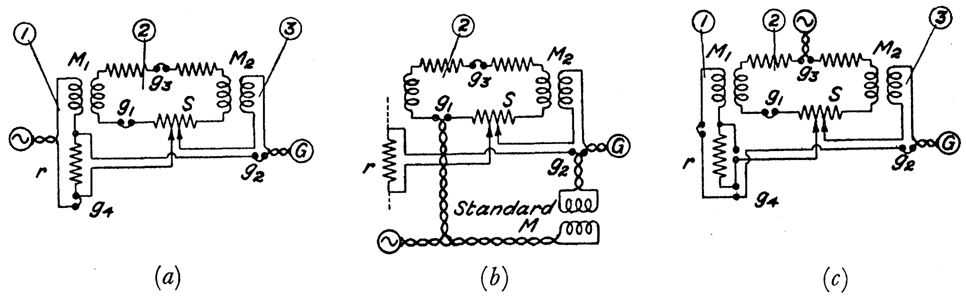

FIG. 7—(*a*) Main bridge : gaps g_1, g_2, g_3, g_4 all closed. (*b*) Comparison of M_2 with standard M. Mesh 1 open, leads inserted at g_1 and g_2. (*c*) Comparison of M_1 with M_2, leads inserted at g_3 and g_4.

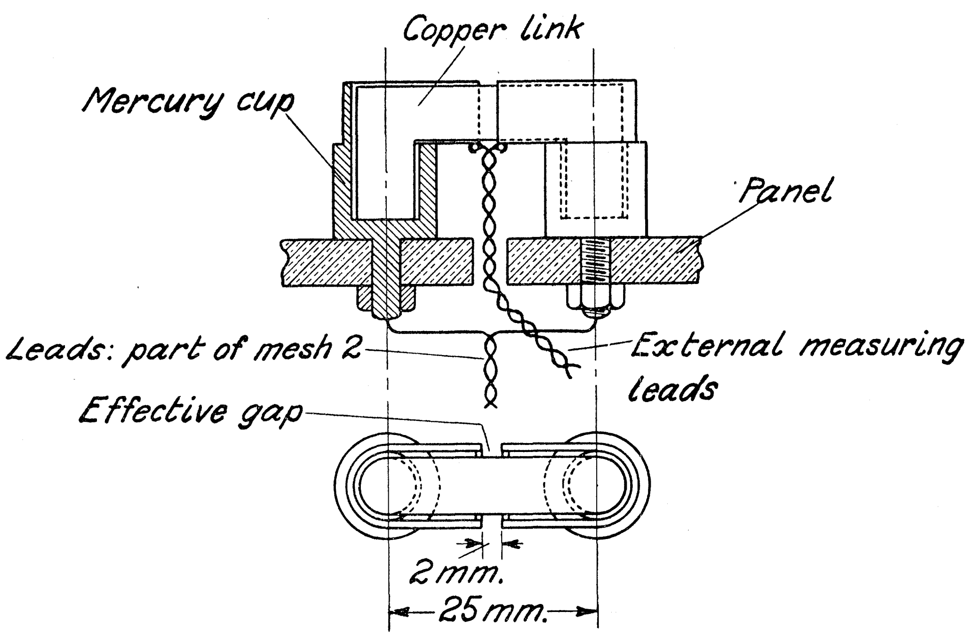

FIG. 8—Duplication of leads for auxiliary measurements.

connexions in meshes (1) and (3) are most conveniently made by the compact type of key used on telephone switchboards, but the condition of stability of resistance of mesh (2) demands the use of mercury cups and copper links of large cross-section for the making and breaking of this circuit. The removal of a copper link from mercury cups of the kind usually employed in resistance-bridges would cause a

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considerable change in the contour of the circuit and therefore of mutual inductance, but by the use of the special form of mercury cups shown in section and plan in fig. 8, the change of contour on removing the link is made completely negligible. This arrangement was employed for mesh (2).

The use of closely twisted leads, admirable from the point of view of suppressing stray inductive effects, has the disadvantage that the comparatively large capacitances of such leads are in parallel with any instrument to which they are connected. This, in the case of the inductive coils, may produce considerable changes in effective mutual inductance and it is necessary, therefore, to arrange that these leads-capacitances shall either not be changed, or changed by determinable amounts, during the auxiliary measurements of M_1 and M_2. Care was taken to satisfy these conditions in the measurements of the frequency characteristics of M_1 and M_2 (*see* section 8).

Having thus outlined the more important practical requirements of the assembly, the description of the actual bridge can be followed more easily. The whole bridge is built up on a marble base, about $100 \times 40 \times 3$ cm. At one end, the inductor M_1 is mounted, with its axis horizontal, on a framework which allows it to be rotated about horizontal and vertical axes and so brought into a conjugate position with respect to M_2, which is mounted with its axis vertical at the other end of the base. Each inductor is covered by a keramot panel, carrying mercury cups with terminals at the ends of the secondary windings (*i.e.*, those included in mesh (2)), and terminals at the ends of the primary windings. The panel over M_1 carries the handles by which the positional adjustments are made, and that over M_2 carries the pointer and scale of the variable part, the moving coil being carried on a mounting bolted on to the upper cheek of M_2. With the exception, of course, of the terminals, all the fittings, including the adjusting gear on M_1, are made from keramot.

Between these two panels rests a third panel (Fig. 9) carrying the remainder of the network. At each end is a pair of mercury cups corresponding to those terminating the secondary windings of M_1 and M_2 and connexion between these corresponding pairs is effected by suitable copper links. The secondary mesh (2), including the ballast-resistance and the potentiometer S, is completed between these mercury cups. The current and potential terminals of S are brought to the top of the panel, the former being linked into mesh (2) by mercury cups and links. The gaps g_1 and g_3 are also brought to the top of the panel with the arrangement shown in fig. 8, and an additional cup and link arrangement in conjunction with the first gap facilitates the connexion of leads for measurements of the resistance of R. At each end of this panel there are also terminals corresponding with the terminals of the primary windings of M_1 and M_2 and these are linked with copper straps. The terminals corresponding to the primary of M_1 are in series with the current terminals of r (which is mounted on the panel), the current supply, and the primary winding of the zero adjuster. Those corresponding to the primary of M_2 are in series with the detector, and the potential terminals of S, r, and the zero adjuster. The zero adjuster itself is mounted under the panel roughly coaxial with M_1, and the operating handle and the potential terminals are brought to the top of the panel. Along the front edge of the panel is arranged a bank of telephone keys, controlling the various circuits. Fig. 9 shows diagrammatically the arrangement of the circuits and, as far as is consistent with the limitations of the drawing, those leads which are in fact twisted closely together are shown side by side. The internal circuits of S and of the zero adjuster are, for simplicity, shown purely formally in this figure.

The whole assembly is immersed in a stoneware tank containing transformer oil, so that all the inductive coils and resistances are protected from the atmosphere and are maintained at a uniform temperature. The material of which the tank was made was non-magnetic, although, since all the inductances in the bridge were measured *in situ* against the primary standard, this requirement is by no means so critical here as for materials used in the immediate neighbourhood of the primary standard itself.

The earthing arms used for the main bridge and the mutual inductance comparisons (*see* next section) are external to the above arrangement, and terminals are provided to which they may be connected. Other external fittings include a resistance box and variable condenser shunting the ballast resistance. The former gives a coarse adjustment of R and the latter a very fine adjustment of L (*see* equation (2.2)), which may be regarded as supplementary to the adjustment of S. This condenser is shown as C in fig. 2.

The tuning-fork oscillator and amplifiers providing the supply current are housed in a metal cabinet some 50 ft. from the bridge, and are connected to it by twinned cable enclosed in an earth-connected sheath of copper braid. The amplifier is actually housed in an outbuilding of the Laboratory some 150 metres from the room in which the bridge stands, and is connected to the bridge through a toroidal transformer, the windings of which are electrostatically screened from one another. The purpose of this transformer is to avoid imposing on the detector points of the bridge the very large earth-capacitances of the long leads to the amplifier, for, although the use of the Wagner earthing-arm renders the effect of these capacitances nugatory, the ease with which the necessary conditions of balance, both on the main bridge and on the Wagner arm, can be approached is largely influenced by the magnitude of the earth-capacitances of the detector arm. In fact, if these capacitances are so large that their impedances become comparable with those in the bridge network, the adjustments on the bridge and on the Wagner arm cease to be independent and the approach to the final balances can be made only by a great number of very small adjustments. By having an isolating transformer near the bridge, all this trouble is avoided, as the earth capacitances which are effective are only those associated with the leads between bridge and transformer, which can be kept very short. The return leads from the amplifier to the galvanometer are in any case isolated from the bridge and their length and associated capacitances are of no very great significance.

With this disposition of the equipment, the stray coupling, both capacitive and inductive, between the bridge, supply, and detector, is reduced to a very low order.

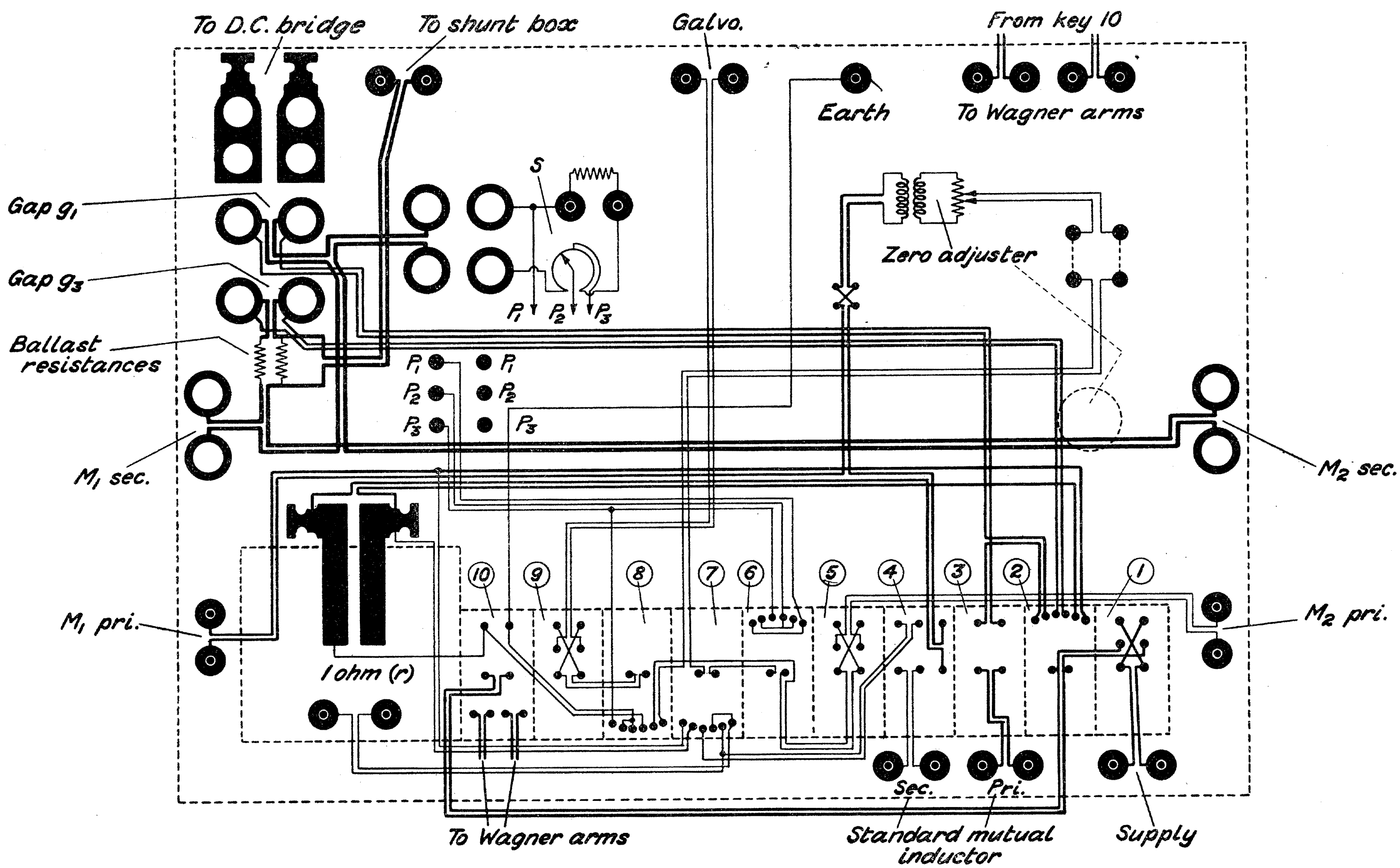

FIG. 9—Diagram of wiring of centre panel. The keys are :—No. 1. Supply : on—off—reverse ; 2. Supply : distribution ; 3. Standard mutual inductor, pri. ; 4. Standard mutual inductor, sec. ; 5. M_2 : pri.—reverse ; 6. S : reverse—centretap ; 7. Galvo, distribution ; 8. Wagner key ; 9. Galvo, on—off—reverse ; 10. Wagner arms or direct earth.

The reversal of the connexions to supply and detector under the best conditions does not produce a change in balance point of as much as 1 part in 10^5 of M_2.

7—Comparisons with Primary Standards

In order to obtain our final result, the values of the bridge inductances M_1 and M_2 must be determined in terms of the primary standard of mutual inductance, and the frequency of the supply current must be measured in terms of the standard second. In addition, the values of the bridge resistances must be measured in " international " units by comparing them with the working standards of the Laboratory. These measurements can now be considered in further detail under their appropriate headings.

(a) *Measurement of* M_1 *and* M_2

The fundamentals of the methods of determining M_1 and M_2 have been outlined in the preceding section. M_2 is first compared directly with the primary standard, the connexion of the latter to the bridge circuit being made through two keys (Nos. 3 and 4, fig. 9). Suitable keys (Nos. 2 and 7, fig. 9) connect the supply and detector in the required positions, giving the arrangement shown in fig. 7*b*.* The complete details concerning the use of the primary standard of mutual inductance are given elsewhere (Campbell, 1907, 1912; Dye and Hartshorn, 1927), but it is desirable to stress here the more important features of the measurement.

In the first place, the primary standard possesses, in virtue of its characteristic design, a secondary coil of many closely-wound layers and therefore of large inductance and high self capacitance. The properties of the instrument are in consequence dependent to a marked extent on the frequency at which it is used, and it has been found that in order that the difference between the calculated value, deduced from its dimensions, and the effective value shall not exceed 1 part in 10^6, the working frequency should not exceed 10 cycles per second. Alternating current of this frequency, supplied from a small alternator set, is therefore used in the comparison of M_2 with the primary standard. It follows, therefore, that the difference, if any, between the effective value of M_2 at this frequency and that at the working frequency of the main bridge (100 cycles per second) must be determined.

The leads to the primary standard are concentric, the outer conductors being connected to the potentiometer (S, fig. 7*b*) common to the supply and detector circuits, and are therefore approximately all at one potential. No capacity currents flow, therefore, between these conductors. A small correction, amounting to about 1 part in 10^6, is applied to the calculated value of the standard to allow for the effects of the capacitances between the inner and outer conductors which act in

* These keys also automatically isolate the primary windings of M_1 (*see* fig. 7*b*) during the comparison of M_2 with the standard. This avoids the imposition of certain capacitances across mesh (2) which are not effective in the main bridge circuit.

shunt across the coils of the instrument. These capacitances were measured separately. The concentric leads are of course astatic and make no direct contribution to the mutual inductance of the circuits under consideration. However, a small mutual inductance (of the order of 0·02 μH, or 2 parts in 10^6 of M_2) is contributed by the keys and associated leads with which the primary standard is connected into the main bridge, and an appropriate correction is applied. These small inductances are easily measured by direct comparison with the variable part of M_2, which can be used separately.

Finally, it is necessary to correct for the effect of capacitances to earth in the comparison bridge. We may regard the effects of capacitance to earth as being represented by the condensers C_1–C_5 shown in fig. 10, in which a potentiometer with its tapping point earthed, is shown connected across the source of supply. With this arrangement C_1 and C_3 are at once disposed of as they fall outside the actual measuring circuit. So far as C_2 and C_4 are concerned, it can be shown that if R_1 and R_2 are the resistances of the primary windings, S_1 S_2 those of the secondary windings, and P_1 and P_2 those of the two parts of the potentiometer on either side of the tapping point, then at low frequencies

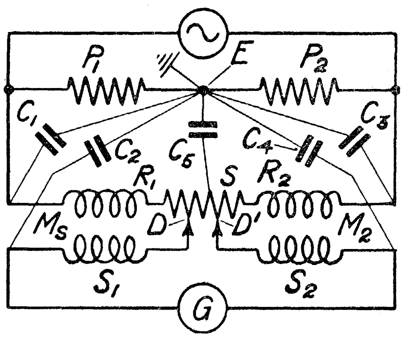

FIG. 10—Earth-capacitance effects in mutual inductance bridge.

$$M_s+M_2+(S_1C_2-S_2C_4)(R_1P_2-R_2P_1)/(P_1+P_2) = 0. \quad (7.1)$$

The effective values of the mutual inductances are therefore modified by the earth-capacitances C_2 and C_4 associated with the secondary or detector circuit, but by arranging that $R_1P_2 = R_2P_1$, the term involving these capacitances vanishes. This can be done easily by connecting a battery in place of the a.c. supply and a galvanometer between E and D (or D′)* and adjusting the tapping point E for balance. In practice it is simpler and sufficiently accurate to retain the a.c. supply and to connect the detector between E and D and to adjust for a minimum deflexion, the operation then being analogous to the balancing of the Wagner arm on the main bridge. This automatically then reduces to zero the potential difference across C_5 which therefore ceases to have any effect on the circuit. Neglect of this precaution may give rise to errors of 2 or 3 parts in 10^5 in the measurement of M_2. A key (No. 10, fig. 9) is used to connect across the current supply either the main Wagner arm or the potentiometer described above, and another key (No. 8, fig. 9) connects the detector in the positions required for the main balances, and for the "earth" balances. The detector used for the measurements at 10 cycles per second is a vibration galvanometer of the Campbell type.

In the comparison of M_2 with the standard, allowance has to be made for the small mutual inductances between the primary of M_2 and the secondary of the

* The resistance between DD′ is of the order of 10^{-4} ohm, so that we may regard these points as at the same potential for the present purpose.

standard, and between the secondary of M_2 and the primary of the standard. If after balance has been obtained, the connexions to the primary and secondary windings of the standard are reversed, then the sign of these stray mutual inductances will be reversed relative to that of the standard inductance, and a new balance point will be found. It is easy to see that the mean of these two balance points will be correct, and that in this way the effect of the stray mutual inductances can be eliminated.

The comparison of M_1 with M_2 is a more straightforward affair, since the stray mutual inductances between the primary of one and the secondary of the other, and *vice versa*, are now actually part of the required inductances and must not, therefore, be eliminated, for, as we have already seen, M_1 and M_2 are the inductances between mesh (1) and mesh (3) respectively, and the whole of mesh (2), which includes both secondary windings. There are, for the same reason, no corrections for the inductance of leads.

The same earthing arrangement is adopted in the comparison of M_1 and M_2 as in the comparison of M_2 with the standard, although in this case the comparatively small resistances in the detector circuit make the adjustment less critical. The comparisons between M_1 and M_2 may be made either at 10 cycles per second or at the working frequency (100 cycles per second) of the bridge; in the first case a knowledge of the frequency corrections of both M_1 and M_2 is required; in the second, of M_2 only. Both methods were used and excellent agreement obtained, giving a satisfactory check on the method of measurement and on the determination of the correcting terms.

It should be mentioned that the positions of the gaps g_1, g_2, g_3, g_4 (Fig. 7*a*, *b*, *c*), which are made in the various meshes of the network in order to carry out these measurements of mutual inductance, were carefully chosen so as to ensure that the conditions as to common point on the windings of M_1 and M_2 were the same for the main bridge and the auxiliary measurements of M_1 and M_2, and that the distribution of capacitance was also the same for the various measurements. In particular the gap g_3 is placed in the middle of the ballast resistor in mesh (2) so as to obtain a symmetrical circuit for the comparison of the two similar inductors M_1 and M_2. The actual positions of gaps g_1 and g_3 are those shown in fig. 7, and a comparison of the various arrangements will show how the appropriate conditions are kept constant from one measurement to another.

(*b*) *The Measurement of Frequency*

The most convenient and accurate way of referring the frequency of the supply current back to the standard second is to make use of the Laboratory frequency standard, which is a valve-maintained tuning-fork, vibrating at 1000 cycles per second and checked continuously against astronomical time signals. This standard, which was designed by the late Dr. D. W. Dye (Dye and Essen, 1934), provides a frequency which is maintained and known to an accuracy approaching 1 part in 10^7.

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In order to compare the frequency of the current supplied to the bridge, which may be 100 cycles per second or 50 cycles per second, with the standard frequency, use is made of a frequency-multiplying circuit (a multivibrator of the Abraham-Bloch type followed by a resonant circuit) by means of which the frequency of the supply current is "stepped up" to about 1000 cycles per second. In essence this process consists in selecting and amplifying the 10th (or 20th) harmonic of the supply current. Beats are then produced with the standard frequency of 1000 cycles per second and are automatically recorded by an electro-mechanical counting device. A complete description of this equipment has been published elsewhere (ASTBURY, 1935), and we shall only concern ourselves here with those of its properties which are of importance for the measurement now under discussion. The apparatus may be considered as consisting of two parts—the beat-producing circuits and the recording circuits. The former are running continuously, but the latter are controlled by the observer, by means of a single switch. Immediately before a reading of the balance point is taken on the bridge, the observer opens this switch and the number of beats occurring and the corresponding time-intervals are then recorded by the apparatus, the process continuing until the readings of the balance-point having been concluded the switch is closed. The starting and stopping circuits are arranged so that the time-interval recorded is always an integral multiple of the beat period, the operation of the counting and timing gear being controlled ultimately by the beats themselves. The use of the apparatus therefore adds nothing to the burden of the observer, since the operation of the circuit is quite independent of the instant he closes or opens the control-switch.

The actual determination of the balance point of the bridge may take 100–200 seconds, which therefore represents the order of the time-interval recorded on the beat-counter. It is easily shown that if f_1 be the supply frequency, f_2 the standard frequency, t the time-interval, and N the beat frequency, then

$$\frac{\delta f_1}{f_1} = \frac{N}{f_2}\frac{\delta t}{t}, \qquad (7.5)$$

where δf_1 is the error in the value of f_1 corresponding to an error δt in the observed value of t. Thus if $f_2 = 1000$ and N is of the order of 1, then if $\delta f_1/f_1$ is to be of the order of 10^{-6}, $\delta t/t$ is of the order of 10^{-3}—in other words, to obtain an accuracy of one part in 10^6 in the measurement of frequency the time interval of 100–200 seconds must be measured with an accuracy of 1 part in 1000. This can be done easily with a good quality stop-clock, and, of course, when N is less than 1 (often it was of the order 0·2) the accuracy required in the timing is correspondingly reduced.

This method of measuring frequency therefore has the great advantage that the working frequency is determined with an uncertainty probably less than 1 part in 10^6 at the same time as the circuit is balanced, and the time-interval required is so short that the readings of frequency and balance-point are sensibly simultaneous.

(c) The Direct Current Resistance Measurements

Having determined M_1, M_2, and ω by the methods outlined above, we deduce therefrom the value of the product Rr in absolute units. We have then to determine the value of this product in International Units, a process which involves the comparison of r and R with the appropriate Laboratory standards. The resistance r is, as has been pointed out, a self-contained standard, and its measurement in terms of the Laboratory standards is made by the well-known methods described in GLAZEBROOK's Dictionary of Applied Physics (1922, p. 717). The resistance R, on the other hand, is that of a highly inductive circuit, part of which is of copper, and which is not in any sense a transportable unit. It is necessary, therefore, to determine R *in situ* and to keep it under continuous observation during the measurements on the main bridge. These measurements necessitate the use of leads between R and the resistance-measuring circuit, but by making use of the principle of the

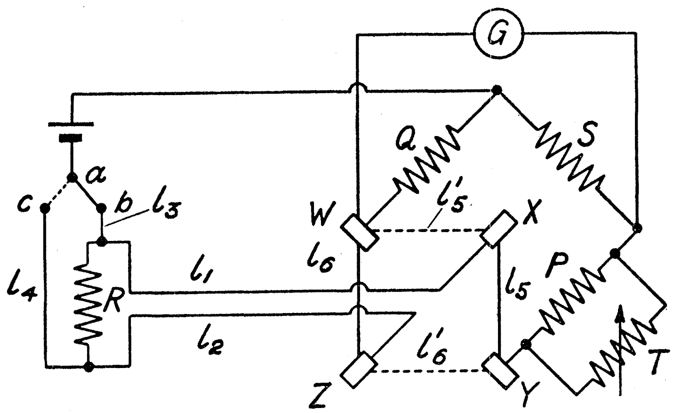

FIG. 11—Smith Bridge for the measurement of R.

Smith bridge (SMITH, 1912), it becomes a simple matter to compensate for the resistance of these leads.

The arrangement used is shown diagrammatically in fig. 11, in which R represents the resistance to be measured. Four leads, l_1, l_2, l_3, and l_4, connect R with the measuring circuit consisting of a pair of ratio arms, Q, S, and a standard resistance P, across which is a shunt box, T, providing the necessary adjustment. The leads l_1 and l_2 terminate in copper blocks, X and Z, containing mercury cups. Similar blocks, W, Y, together with a pair of links l_5, l_6, complete a change-over switch. The leads l_3, l_4, terminate on a switch, a, b, c, in the battery circuit. The bridge may be balanced with the links l_5 and l_6 in the position shown by the full lines in fig. 11, and with a joined to b. We then have, if P_1 be the effective value of the combination of P and T,

$$R + l_2 + l_6 = \frac{Q}{S}(P_1 + l_1 + l_5). \quad \quad (7.6)$$

The links l_5, l_6 are then moved into the positions indicated by l_5', l_6', and a is joined

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to c instead of to b. The bridge is rebalanced by adjusting T, and, if P_2 be the new effective value of the combination of P and T,

$$R + l_1 + l_5 = \frac{Q}{S}(P_2 + l_2 + l_6). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (7.7)$$

The ratio Q/S is chosen to be nearly unity and we may represent it by $1 + \alpha$, where α is of the first order of smallness. We therefore find, from the above equations,

$$2R = (1 + \alpha)(P_1 + P_2) + \alpha(l_1 + l_2 + l_5 + l_6). \quad . \quad . \quad . \quad . \quad . \qquad (7.8)$$

If the ratio coils be interchanged, two similar balances, giving values P_3 and P_4, can be obtained, and we have

$$2R = (1 - \alpha)(P_3 + P_4) - \alpha(l_1 + l_2 + l_5 + l_6), \quad . \quad . \quad . \quad . \qquad (7.9)$$

giving

$$R = \tfrac{1}{4}(P_1 + P_2 + P_3 + P_4) + \tfrac{1}{4}\alpha(P_1 + P_2 - P_3 - P_4). \quad . \quad . \quad . \qquad (7.10)$$

It is easy to show that the second term in this expression is approximately equal to α^2R, so that even if α is as great as 10^{-4}, the value of R will be equal to the mean value of the four values of P with an accuracy of 1 part in 10^8.

It will be evident that since the links l_1 and l_2 are included in the bridge arms, they must be of low, constant, and, if possible, equal resistances. These conditions are easily secured by making the links of copper rod, of diameter 1 cm., which make contact with the copper blocks by means of mercury cups. It will also be apparent that the points at which l_3 and l_4 are attached to R must be the actual points between which the resistance is required. In other words, the leads l_1 and l_2 are analogous to the "current" leads, and the leads l_3 and l_4 to the "potential" leads of a four-terminal shunt.

Since the value of R is about 40 ohms for the measurements at 100 cycles per second, it was convenient to make the standard P of a special form. It consists of two equal 20-ohm resistances which can be connected in series or in parallel by a suitable arrangement of copper links and mercury cups. In the first case the resistance value is 40 ohms and therefore suitable for use in the bridge described above, and in the second case its value is 10 ohms and is therefore readily comparable with the Laboratory standards. By a well-known result its value in the first case is exactly four times its value in the second. For the 50-cycle measurements, R is 10 ohms and P is conveniently provided by a Laboratory standard.

The ratio arms and standard resistance P are immersed in a tank of transformer oil thermostatically controlled at 20° C., and the switching arrangement for altering the position of the links l_5 and l_6 and the switch a, b, c is conveniently arranged on one handle so that the complete change-over may be made in one movement. The high inductance of the resistance R makes impracticable the method of having the detector circuit permanently closed and testing for balance with a tapping key in the battery circuit, and a special six-position control switch was constructed, operating according to the schedule in Table II.

TABLE II

Position	Battery Circuit	Galvanometer Circuit	Adjustments
1	Open	Open	—
2	Open	Closed	Adjust scale zero
3	Open	Open	—
4	Closed	Open	—
5	Closed	Closed	Balance bridge
6	Closed	Open	—

The use of this switch made it possible to carry out the operations quickly without any risk of throwing the galvanometer out of adjustment accidentally by a violent "inductive kick". In this way a close watch was kept on the scale zero and on thermal effects in the circuit. It is estimated that the measurement of R by this method was not in error by more than one or two parts in 10^6.

In order to make this measurement of R, circuit (2) of the Campbell bridge must be opened by the removal of one of the copper links previously described. The resistance of this link must be added to the observed value, and this was done although the correction was in most cases negligible.

8—The Determination of the Correcting Terms

The measurements discussed in the preceding section are, of course, carried out in their entirety in every experimental determination, and we have now briefly to consider the non-recurrent measurements leading to the determination of the correction factor ψ given in equation (3.13). This may be conveniently re-written here :—

$$\psi = \frac{1}{M_1 M_2}\left[(Ll - M_1\lambda) - \frac{rR + S\sigma_1}{\omega^2}\right]. \quad \ldots \ldots \quad (3.13)$$

In view of what was said on p. 437 (*see* equation (4.22)), it is evident that for λ in this expression we must write the "differential" value $(\lambda_1 - \lambda_0)$ in which λ_1 is the residual inductance of the potentiometer at the setting used in the main bridge balance and λ_0 that at the setting used in the mutual inductance bridges. Further, the method of comparison of M_2 and M_1 with the primary standard requires that we shall know the quantities ΔM_1, ΔM_2, such that

$$\Delta M_{1,2} = \text{effective value of } M_{1,2} \text{ at 100 c/s.} - \text{effective value of } M_{1,2} \text{ at 10 c/s.}$$

The measurement of the residual inductances l and λ is a piece of routine calibration which has been described fully elsewhere (HARTSHORN, 1927 ; ASTBURY, 1931).

The quantities ΔM_1, ΔM_2 and σ_1 are measured by comparing the inductances concerned with a special standard of mutual inductance, the properties of which are accurately deducible from a series of measurements at radio-frequencies from which the capacitances of the system are found (HARTSHORN, 1926). The quantity ΔR, the amount by which the resistance of the secondary mesh at 100 cycles per second exceeds its resistance to direct current, is determined by a comparison of the circuit with a high-frequency inductance standard, the properties of which are also determinable from radio-frequency measurements.

The corrections due to l and λ do not vary with the connexions of the inductive coils. The quantities ΔM_1, ΔM_2, ΔR, and σ_1 differ, however, by small amounts for the eight connexions used (*see* p. 433). This is due to the fact that they are largely determined by the capacitances of the system, which alter with the different connexions. These four quantities are also functions of the frequency, varying very nearly as the square of the frequency. The corrections are very much larger, and so determinable with correspondingly higher accuracy, at higher frequencies. Measurements of these quantities were made, therefore, not only at the working frequency but also over a range of audio-frequencies, and curves were drawn showing the various corrections as functions of frequency. The values corresponding to the working frequency were then read from these curves.

Table III gives the values of the various correcting terms, together with the contribution made by them to the final result (*i.e.*, the ratio of the international to the absolute ohm). The corrections are given without regard to sign, the purpose of the table being to indicate only the magnitude of the quantities involved.

TABLE III—CORRECTING TERMS

Quantity	Mean value for the 8 corrections at 100 c/s.	Contribution to result, *ppm.*
l	$0\cdot01_7$ μH	1
$\lambda_1 - \lambda_0$	$0\cdot33$ μH	17
σ_1	$2\cdot5 \times 10^{-4}$ ohm	3
ΔR	$3\cdot0 \times 10^{-4}$ ohm	4
ΔM_1	$0\cdot10$ μH	5
ΔM_2	$0\cdot06$ μH	6

Actually, since some of the terms have opposing effects, the total correction applied to the observed quantities amounted on an average to 10 or 20 parts per million. It is estimated that this correction is not in error by more than 10%.

9—EXPERIMENTAL PROCEDURE

In order to obtain the highest precision, it is important that the whole sequence of observations, including the balancing of the Campbell bridge, the measurements of mutual inductance, frequency, d.c. resistance, etc., should be made in a short time, otherwise some of the quantities may have varied appreciably by the time the remaining ones have been measured. It is also desirable that one observer should be able to take all the readings required. All these conditions are satisfied by the circuit arrangements already described. None of the wiring of the circuits is disturbed for any of the measurements: the observer makes all the necessary changes either by the operation of switches or by the removal of copper links from pairs of mercury cups. Tables showing the appropriate positions of the switches and links are prepared in advance and the observer is able to devote his whole attention during an experiment to the taking of readings with the highest accuracy.

Full consideration was given to the effects produced by the changes made in the network at the time the wiring of the switches was carried out. At this time many subsidiary observations were made of the differences obtained when given changes were produced by the actual transfer of wires instead of by the operation of switches, and the use of any switch that seemed likely to have any undesirable effect on the capacitance or inductance of the system was checked in this way.

The actual course of a determination was as follows:

(*a*) *Adjustment of Current*

The first adjustment required before the commencement of a series of observations is that of the currents in the a.c. and d.c. bridges. For obvious reasons, the currents in the secondary mesh should be approximately equal in the two cases. If the current in mesh (1) in the Campbell bridge be i_1, then the current i_2 in the secondary mesh (2) is given by the simple relation

$$i_2 = i_1/(1 + R/r)^{\frac{1}{2}}, \qquad (9.1)$$

since $L \doteqdot M_1 \doteqdot M_2 \doteqdot Rr/\omega^2$. The current supplied to the d.c. bridge used for measuring R must clearly be $2i_2$.

(*b*) *Preliminary Balance of Bridge*

The required arrangement of links and keys having been made, a preliminary balance is obtained on the Campbell bridge, the shunt box across the ballast resistance in mesh (2) being adjusted until a convenient reading is obtained on the scale of M_2, the balance-point of the Wagner arm also being found. A preliminary balance on the d.c. bridge is also obtained.

(*c*) *Zero-adjustment*

The secondary mesh is then opened by the removal of the links connecting the secondary windings of M_1 and M_2 and the potential leads of r are both connected to one terminal (F in fig. 2). The mutual impedance between meshes (1) and (3) is then reduced to zero by adjusting the position of the mutual inductance M_1 by the mechanical arrangements previously described (p. 448) and by the zero-adjusting potentiometer. For this, as indeed for all balances on the a.c. network, four settings are taken corresponding to the four combinations resulting from the reversal of the switches to the supply and detector. In general, the differences between the four corresponding readings on the zero adjustments are very slight and the controls are set in the mean positions for the subsequent work on the main bridge.

(d) *Measurement of* M_2 *and* M_1

The switches and links are then set so as to arrange the circuit for the comparison of M_2 with the standard, a process requiring eight observations, corresponding to the four combinations of supply and detector leads referred to above, and the two possible connexions of the primary and secondary windings of the standard (*see* p. 453). The comparison between M_1 and M_2 then follows.

(e) *Operation of the Main A.C. and D.C. Bridges and of the Frequency Measuring Equipment*

The temperature of the tank containing the main bridge is not controlled, but the large volume of oil gives the system a thermal stability quite as great as that obtained by the ordinary methods of control. The standard resistor r is, of course, immersed in this tank, and its temperature-coefficient of resistance has been accurately determined, so that the value under working conditions can be related without sensible error to its value determined by comparison with the Laboratory standards, a measurement which is always carried out at 20° C. The temperature in the main tank is therefore recorded immediately before the commencement of the next stage in the experimental procedure, which is the determination of M_2, ω, and R under the conditions of balance of the Campbell network.

The balance of the main bridge is then obtained, but not recorded, the purpose of this being to fix the resistance of the shunt box, and therefore of R, for the whole series. R is then measured on the d.c. bridge, and then the beat-counting gear is started and runs during the recording of the balance-point of the main bridge, an operation requiring four readings as discussed above. The a.c. balance having been obtained, the beat-recording gear is switched off and the number of beats and time-interval noted. The measurement of R is then repeated. This sequence is repeated four or five times, the whole occupying about 20 or 30 minutes, the temperature of the tank being recorded at regular intervals.

(f) *Concluding Measurements of* M_1 *and* M_2

After the final measurement of R, the mutual inductance values are again checked.

The overall variations of the recorded quantities during a complete series of observations are recorded in Table IV.

In very few cases did the variations exceed those given in the above table, so that the stability of balance of the whole system is almost of the same order as the observational error. The comparison of r and of the 10–40 ohm standard used for the measurement of R with the Laboratory standard resistance coils was made on alternate days during the progress of the measurements.

The current supplied to the Campbell bridge was of the order 200–250 ma., and the sensitivity was such that a displacement of balance-point corresponding to a change of 1 part in 10^5 of M_2 produced a galvanometer deflexion of about 15 mm. at a working frequency of 100 cycles per second. With a frequency of 50 cycles per second the arrangements were slightly less sensitive and a deflexion of about 10 mm. was obtained for a change of 1 in 10^5 on M_2. The sensitivity in the mutual inductance comparisons was of the same order, so that in all the arrangements of the a.c. networks a change in balance-point corresponding to an alteration of 1 part in a million of M_2 was actually easily visible. On the d.c. bridge used for the measurement of R, a change in R amounting to 4 parts in 10^7 was just detectable.

Table IV—Variation in Observed Quantities during Observations

Quantity	Variation
Temperature of Tank (r)	0·1° C. (1 part in 10^6 on r).
Scale reading of M_2: comparison with standard	0·01 μH (1 part in 10^6).
Scale reading of M_2: comparison with M_1	0·00 μH.
Scale reading of M_2: main bridge balance	0·02 μH (2 parts in 10^6).
Frequency	1 part in 10^6.
R	2 parts in 10^6.

So far as the work at 100 cycles per second was concerned, the working conditions were uniformly very satisfactory. The interference from power cables in the vicinity was barely noticeable, and the only really troublesome disturbances were mechanical in origin. The operation of heavy machinery in the neighbourhood, or anything likely to cause vibrations of the building, slightly disturbed the tuning-fork and therefore the frequency of the current, thus causing an irregular disturbance of the balance. On the whole, however, any such disturbances quickly died down and did not cause any serious trouble. On the other hand, the measurements at 50 cycles per second were often made tedious and difficult by electrical disturbances of obscure origin. Similar troubles are usually experienced when precision measurements are carried out at the frequency of the ordinary power supply, but in the present case the effect on the accuracy was scarcely appreciable.

10—Results

It is not feasible to print in detail all the actual experimental observations, but a record of a typical set is included to illustrate the routine described in the previous section.

Observations made 9 June, 1936

Arrangement of Links—Set 1

(Ref. E.372/112)

1.—*Comparison of* M_2 *with* M_S (*the Primary Standard*) *at* 10 *cycles per second*

	Before main bridge readings. M_2 (tank) at 15·2° C. : M_S at 15·4° C. I	II	After main bridge readings M_2 at 15·2° C : M_S at 15·4° C. I	II
1	−1·53 μH*	+0·16 μH	+0·16 μH	−1·53 μH
2	−1·56	+0·11	+0·11	−1·56
3	−1·50	+0·17	+0·16	−1·50
4	−1·48	+0·24	+0·25	−1·46
Means	−1·52	+0·17	+0·17	−1·51
	−0·67		−0·67	

Mean reading corrected for scale calibration = − 0·64 μH.

2—*Comparison of* M_1 *with* M_2

	Before main bridge readings. At 10 c/s.	At 100 c/s.	After main bridge readings. At 10 c/s.	At 100 c/s.
1	3·60 μH	3·40 μH	3·60 μH	3·42 μH
2	3·54	3·36	3·55	3·37
3	3·54	3·35	3·55	3·37
4	3·60	3·35	3·60	3·37
Means	3·57	3·37	3·57	3·38

Mean readings corrected for scale calibration : At 10 cycles, 3·54 μH ; at 100 cycles, 3·35 μH.

* The readings given in μH refer to the variable part (scale) of M_2. The rows marked 1, 2, 3, 4 refer to the four combinations of the supplyand detector keys (*see* p. 459) : the columns I and II refer to the two methods of connexion of M_S (*see* page 453).

3—*Main Bridge and D.C. Bridge Readings, and Measurement of Frequency. Current in Campbell Bridge*, 205 *m.a.*

Temperatures : Main tank, 15·2° C. ; d.c. tank, 20·0° C.

M_2 scale readings		Time	Beats	Shunts (D.C. bridge)	
—		—	—	3 008·8	3 002·9
—		—	—	3 008·3	
—		—	—	2 997·2	
—		—	—	2 997·5	
1·39	μH 1·32	0	8 170	2 997·5	3 002·7
1·25		100·6 secs.	8 217	2 997·1	
1·34			47	3 007·9	
1·29		Δf† = 47/1 006 = 0·0467$_2$ c/s.		3 088·2	

† Δf = Actual frequency — 100·0000 cycles per second. The correction for the rate of the clock used to time the beats was less than 1 part in 1 000.

M_2 scale readings		Time	Beats	Shunts (D.C. bridge).	
1·37	1·31	0	8 217	3 008·1	3 002·4
1·25		103·0 secs.	8 265	3 007·6	
1·34			48	2 996·8	
1·29		$\Delta f = 48/1\,030 = 0{\cdot}0466_0$ c/s.		2 997·1	
1·36	1·30	0	8 265	2 997·2	3 002·4
1·25		109·2 secs.	8 316	2 996·9	
1·31			51	3 007·6	
1·29		$\Delta f = 51/1\,092 = 0{\cdot}0467_0$ c/s.		3 007·9	
1·36	1·31	0	8·316	3 008·1	3 002·4
1·25		109·4 secs.	8·367	3 007·6	
1·33			51	2 996·9	
1·29		$\Delta f = 51/1\,094 = 0{\cdot}0466_2$ c/s.		997·2	

Therefore mean value of M_2 scale corrected for calibration = 1·28 μH.
Mean value of $\Delta f = 0{\cdot}0466_6$ c/s. Correction for rate of standard 1000 c/s. fork is — 3 parts in 10^7 o that

$$f = 100{\cdot}0466_3 \text{ cycles per second.}$$

Mean value of shunt = 3 002·5 ohms = 3 002·6 ohms corrected for leads and calibration of box.

4—*Calculation of Result*

(i) *Evaluation of* $Rr/4\pi^2$ $(= M_1M_2f^2)$ *in Absolute Units*

M_s = 9 997·27 μH at 18·0° C.
= 9 997·11 μH at 15·4° C. (Temp. Coeff. = $6{\cdot}3 \times 10^{-6}$ per 1° C.)
= 9 997·09 μH at 15·4° C. corrected for leads inductance.

Therefore fixed part of M_2 = 9 997·09 — (— 0·64) = 9 997·73 μH.
M_1 = 9 997·73 + 3·54 = 10, 001·27 μH from the 10 c/s. readings.
= 9 997·73 + 3·35 = 10, 001·08 μH from the 100 c/s. readings.

Also, M_2 for the main bridge balance = 9 997·73 + 1·28 = 9 999·01 μH.
Thus, using the 10 c/s. measurements on M_1, we have

$$M_1M_2f^2 = 10\,001{\cdot}27 \times 9\,999{\cdot}01 \times (100{\cdot}0466_3)^2 = 1{\cdot}000961 \text{ ohms}^2.$$

$$\text{Correction for } \psi,\ \Delta M_1 \text{ and } \Delta M_2 = \underline{\qquad 46}$$

$$\text{Therefore } M_1M_2f^2 \text{ (corrected)} = 1{\cdot}000915 \text{ ohms}^2.$$

Using the 100 c/s. measurements on M_1, we have

$$M_1M_2f^2 = 10\,001{\cdot}08 \times 9\,999{\cdot}01 \times (100{\cdot}0466_3)^2 = 1{\cdot}000942 \text{ ohms}^2.$$

$$\text{Correction for } \psi \text{ and } \Delta M_2 = \underline{\qquad 28}$$

$$\text{Therefore } M_1M_2f^2 \text{ (corrected)} = 1{\cdot}000914 \text{ ohms}^2$$

The two values of $M_1M_2f^2$ therefore agree to 1 part in 10^6.

(ii) *Value of* $Rr/4\pi^2$ *in International Units*

Standard P in Smith bridge = 40·00105 international ohms. Therefore

$$\frac{1}{R} = \frac{1}{3\,002{\cdot}6} + \frac{1}{40{\cdot}00105}\ *$$

whence R = 39·47515 international ohms.

Also, r = 1·000050 international ohms at 20° C.

= 0·999996 ,, ,, 15·2° C.

Therefore $Rr/4\pi^2$ = 39·47515 × 0·999996/39·47842.

= 0·999913 (international ohms)^2.

(iii) *The Ratio of the International Ohm to the Absolute Ohm*

If this ratio is ρ, then

$$\rho^2 = M_1M_2f^2 \div (Rr/4\pi^2) = 1{\cdot}000914/0{\cdot}999913$$

or ρ = 1·000500

i.e., 1 international ohm = 1·000500 absolute ohms.

In Table V are recorded the values obtained for ratio of the N.P.L. International Ohm to the absolute ohm. The values are recorded in octets, corresponding to the sets 1–8 arising from the eight possible arrangements of the inductive coils. The 23 groups of eight are arranged in chronological order, the observations being made in February, March, May, June, September, and October, 1936, and the working frequency is indicated at the head of the column. It will be seen that there is no significant difference between the measurements at 100 cycles per second and 50 cycles per second so far as the final means are concerned, and we may consider all the measurements as being of equal weight.

The values obtained for the different sets show variations of the order of a few parts in a hundred thousand, but the means of the 23 groups show variations of the order of only a few parts in a million.

It follows from the arguments developed in previous sections of the paper that if all the conditions stated have been satisfied and all the corrections properly applied, the result obtained for each of the arrangements of the inductive coils should be the same. The Table of actual results shows clearly that for any one arrangement of the inductive coils, the departures from the stipulated conditions are appreciable. On the other hand, the mean of each group of eight values is constant within the observational error, and from a consideration of all the results obtained with varying conditions the authors conclude that, although departures from the assumed conditions exist for each arrangement of the coils, the signs of the resulting errors are reversed with respect to M_1 and M_2, on reversing the connexions to the coils, in such a manner that the mean of the results obtained with every possible arrangement is sensibly correct.

* No significant error is introduced by using the reciprocal of the mean rather than the mean reciprocal of the shunt values.

It may be recalled that the assumptions made are as follows :

(*a*) The insertion of a twisted pair of leads into a very small gap in any of the circuits, and the operation of the key connected to it, produces no change in the coefficients of mutual inductance of the circuits.

(*b*) The distributed capacitance of the network is not affected by the various changes made during the course of a determination, and may be correctly represented by lumped capacitances attached to the several components.

Undoubtedly, both of these conditions are satisfied to a first approximation, but it is equally certain that they are not rigorously true, and some departure from the ideal result is to be expected.

Some of the variations within each group of eight readings must be due to stray mutual inductance in the leads to the switches, for after groups I to III had been obtained, the wiring was re-examined and certain small loops closed by a re-arrangement of some leads and the replacement of others. An examination of the results of group IV shows that the individual results are changed by this procedure by amounts of almost the same order as deviations from the mean, while the mean value itself is not appreciably altered. After this, considerable rearrangements of the circuits under the centre panel were made, but again the mean result was reproduced with all the accuracy that could be expected in spite of variations in the separate results. There can therefore be little doubt that stray inductances can produce variations of the same order as those observed, but that they are eliminated on taking the mean of the results corresponding to the eight possible arrangements. It is, of course, to be expected that reversals of the connexions to the inductive coils would reverse the relative signs of the main and stray mutual inductances, and that the effects of the latter would vanish on taking the mean of all the possible arrangements.

It is, however, probable that some other factor is also involved, for the separate results obtained at a frequency of 50 cycles per second are different from those obtained at 100 cycles per second, although the mean is the same. There is no reason to suppose that the stray mutual inductances will vary with the frequency, or the resistance of the network, which is the only factor changed at the same time as the frequency, but the effects of distributed capacitance will certainly vary with the change of frequency and resistance. For example, BUTTERWORTH (1921) has shown that the mutual capacitance, C_{12}, of a pair of inductive coils of resistances R_1 and R_2 joined at one end, contributes to their effective mutual inductance an amount $C_{12}R_1R_2$, which will, according to the relative directions of winding of the coils (*i.e.*, sign of the mutual inductance), either augment or diminish the value. This effect, although negligible in most experiments, is not likely to be so in the present circuit since the capacitance between the windings of M_1 and M_2 must be relatively large. If condition (*b*) relating to distributed capacitance were strictly satisfied, this effect would be correctly allowed for, but the inevitable departures from the ideal condition must introduce small errors of this kind. It is easy to show that any such error varying in sign with the sign of M_1 or M_2 will vanish on taking the mean of the eight readings, provided the reversal does not affect the magnitude of the error as well as its sign.

For these reasons it is considered that the mean value of a set of eight readings should be regarded as a single determination. We therefore regard the results given as representing 23 complete determinations of the ratio of the units. The mean deviation of these values from the final mean, 1·000 501, amounts to only 3 parts per million, all the values lying within a range of 15 parts per million.

Table V—Values Obtained for the Ratio N.P.L. International Ohm/C.G.S.Ohm.

Group . . .	I	II	III	IV	V	VI	VII	VIII	IX	X	XI	XII
	100~	100~	100~	100~	100~	100~	100~	100~	100~	100~	100~	50~
Set	1·000											
1 . . .	506	503	507	499	502	500	498	500	500	501	501	505
2 . . .	528	522	536	519	514	515	515	519	525	518	526	502
3 . . .	483	492	501	503	501	499	511	499	496	499	498	493
4 . . .	478	477	470	475	486	481	480	490	493	491	493	498
5 . . .	478	473	474	478	502	495	492	506	493	495	491	496
6 . . .	485	485	487	509	492	497	498	509	499	499	501	488
7 . . .	531	524	529	517	517	491	491	515	509	514	519	502
8 . . .	506	503	500	487	503	515	514	489	495	491	494	534
Means . . .	499	496	500	498	502	499	500	501	501	501	503	503

Group . . .	XIII	XIV	XV	XVI	XVII	XVIII	XIX	XX	XXI	XXII	XXIII	Means
	50~	50~	50~	50~	50~	50~	50~	100~	100~	100~	100~	
Set	1·000											
1 . . .	510	499	508	507	518	497	505	511	503	512	504	504
2 . . .	505	499	507	507	511	498	502	530	522	529	527	516
3 . . .	500	487	488	493	499	491	492	518	506	518	505	494
4 . . .	498	488	486	496	498	490	486	505	496	503	496	488
5 . . .	490	498	495	494	489	504	498	496	501	496	500	492
6 . . .	467	493	473	484	466	498	479	496	504	496	506	491
7 . . .	489	511	501	507	493	511	500	512	521	512	522	510
8 . . .	506	520	514	519	506	525	514	515	521	515	523	509
Means . . .	495	499	497	501	498	502	498	510	509	510	509	501

Mean of 100~ results = 1·000 502 Mean of 50~ results = 1·000 499

Grand mean = 1·000 501

11—Probable Errors

The work involved in the measurement of a resistance by the Campbell method in terms of the units of length and time may for the consideration of probable errors be conveniently divided into two parts.

(*a*) The construction of a standard of mutual inductance, the measurement of its linear dimensions, the comparison of the magnetic permeability of the material of

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the standard with that of a vacuum, and the calculation of the value of the mutual inductance of the standard by the use of the generally accepted equations.

(*b*) The determination of the relation between the resistances, mutual inductances, and frequency for a special circuit, the comparison of those mutual inductances with the standard and of the resistances with one another.

The first part of the work has not been described in this paper, since it was done by methods which have been fully described in previous publications. The standard of mutual inductance used was the original Campbell instrument. The linear dimensions were, however, redetermined and a new secondary coil was constructed and other small modifications introduced, so that the instrument as used was very similar to one which was constructed at the Laboratory and fully described in 1927. The measurements of linear dimensions were made by the methods described in the paper of 1927 (Dye and Hartshorn, 1927) and the maximum probable errors on the calculated value of the standard are estimated to be of the same order as those then given, viz., about 6×10^{-6} arising from uncertainties of diameter of primary coils, 6×10^{-6} for the axial dimensions, and 3×10^{-6} for the cross-section of the secondary coil, making a total of 15 parts per million. The question of magnetic permeability of the material was investigated in more detail than on the previous occasion. When the new secondary coil was constructed samples of the marble were taken, and their permeability measured by means of a modified Curie balance. The samples were found to be diamagnetic with a permeability less than unity by about 1×10^{-5}. Approximately the same value was obtained for several samples of marble of different kinds, including some quite heavily marked by impurities, so that there is no doubt that this value may be accepted for all the marble of the standard. It would, however, be difficult to calculate the effect of this permeability on the value of mutual inductance, but since the marble occupies only a fairly small fraction of the space occupied by the instrument, the effect of the departure from unit permeability can hardly amount to more than 1 or 2 parts per million, and was neglected.

A check on the calculated value of mutual inductance was obtained in the following way. In the Lorenz apparatus, the Laboratory possesses equipment by means of which resistance may be measured in terms of length and time quite independently of the apparatus described in the present paper. The coils and disks of the Lorenz apparatus constitute primary standards of mutual inductance which are of the same general form as the Campbell standard. Their values are also determined in a similar way and with much the same probable error. By using each pair of coils of the Lorenz machine, in conjunction with a special secondary coil, of the same diameter as the disks, but with the number of turns necessary to make the mutual inductance approximately equal to that of the Campbell standard, a comparison can be made, by the method described in this paper, of the Campbell standard and what may be termed the Lorenz standards. In this way we obtain the relative values of three standards each of about the same estimated accuracy, and by taking a mean we arrive at a value for which the probable error is less than that of each taken separately. The result of these comparisons can be expressed in the form

$$1 \text{ Henry by Lorenz Standards} = 1 + 6 \times 10^{-6} \text{ Henry by Campbell Standard.}$$

The experimental errors of these comparisons are of the order of 5 parts per million, and it was not considered advisable to attempt to correct any of the calculated values on the basis of the results. They do, however, show that the uncertainty of the calculated values probably does not exceed 1×10^{-5}, which is therefore our estimate of the maximum probable error of section (*a*) of the work.

With regard to section (*b*), we have shown that the average deviation of the values given by the 23 determinations from the final result is only 3 parts per million, and this in spite of the fact that some of the measurements were made with different frequencies, different values of resistance, and different arrangements of the circuit. Moreover, all the readings required could be taken with an accuracy of 1 part in 10^6. We have given reasons for believing that the systematic errors are eliminated by taking the mean of a group of eight readings obtained by reversing in turn the various components of the network. The only remaining errors requiring consideration are those arising from uncertainties in the small correcting terms. These only amount to 20×10^{-6} or less, and they can all be determined with an accuracy of 10 per cent. or better. We therefore estimate the probable error of the purely electrical measurements as $\pm 5 \times 10^{-6}$.

The total estimated uncertainty of our absolute measurement of resistance therefore amounts to $\pm 15 \times 10^{-6}$, and we write

$$1 \text{ International Ohm (as realized at the N.P.L.)} = 1 \cdot 000\,500 \pm 15 \times 10^{-6} \text{ ohms.}$$

12—Conclusion

It only remains to notice the agreement between the results of our measurements and those of previous absolute measurements of resistance, and to consider the relation between the ohm as determined by our measurements and the International Ohm as defined by the London Conference of 1908. As stated in Section 1, it is doubtful whether the International Ohm has ever been realized with an uncertainty less than 2×10^{-5}, and discrepancies as large as 3×10^{-5} have been noted. It is therefore only possible to compare two absolute determinations of resistance with full accuracy if they have been carried out at the same time and referred to the same resistance standards. It has fortunately been possible to satisfy these conditions at the National Physical Laboratory during 1936. Measurements were made by the Lorenz method as developed by Sir Frank Smith at the same time as those described in this paper. Details of the measurements by the Lorenz method are in course of publication in the Collected Researches of the National Physical Laboratory, but it may be stated that the result was

$$1 \text{ International Ohm (as realized at the N.P.L.)} = 1\cdot000\,50 + 2 \times 10^{5} \text{ ohms.}$$

The two methods therefore agree perfectly within their estimated errors, a most satisfactory state of affairs. An important point in connexion with this agreement is that the two systems of electrical measurements involved are entirely different, for Lorenz method employs direct current technique, while the Campbell method depends entirely on the technique of alternating-current bridge-measurements. It is very satisfactory to have established the fact that the two systems of measurement are consistent to such a high accuracy. The above two results have been stated in terms of the value accepted for the International Ohm at the National Physical Laboratory in 1936. This value was based on measurements made with columns of mercury in 1912, and in 1924, and on assumptions regarding the constancy of certain standard resistance coils. The values obtained for the coils from the experiments with mercury-tubes in 1924 differed by 25 parts per million from those obtained in 1912. As the estimated experimental error was of this order, there was no evidence of change of value of the coils. They were therefore assumed to have remained constant and the mean of the 1912 and 1924 values taken. They have also been assumed to have remained constant since 1924. The relative constancy of coils of different ages over the whole period, together with the measurements with mercury tubes in 1912 and 1924, and absolute measurements in 1914 and 1936 (*see* below), suggest that no important changes of value have occurred, and it is considered that the value accepted in 1936 differed from that of the definition by no more than 20×10^{6}. We may therefore write

$$1 \text{ International Ohm} = 1\cdot000\,500 \pm 35 \times 10^{6} \text{ ohms.}$$

The more recent of the previously published values for this quantity are collected together in Table VI.

TABLE VI

Reference	Laboratory	Date of measurements	Method	Value of International Ohm in Ohms
F. E. SMITH (1914)	N.P.L.*	1914	Lorenz	1·000 52
GRÜNEISEN and GIEBE (1920)	P.T.R.†	1914	Self-Inductance Standard and Maxwell capacitance bridges	1·000 51
CURTIS, MOON, and SPARKS (1936)	N.B.S.‡	1936	Ditto	1·000 45
P. VIGOUREUX (1937)	N.P.L.	1936	Lorenz	1·000 50
Present paper	N.P.L.	1936	Campbell	1·000 50

* National Physical Laboratory.

† Physikalisch-Technische Reichsanstalt.

‡ National Bureau of Standards.

The results obtained by Albert Campbell have not been included in the table since they were not considered to be of comparable accuracy. His early experiments (1912) gave a value for the International Ohm of 1·000 2_6 ohm, and his later experiments (1925) a value of 1·000 5_4. The methods first used were subject to practical and theoretical limitations, which Campbell himself has discussed, and which are quite sufficient to account for the deviation of this result from the others, while the later experiments were only regarded as preliminary with "limits of accuracy perhaps worse than $\pm 0{\cdot}0001$". It is an advantage of the last method, which is that elaborated in the present paper, that the very simple apparatus employed by Campbell is capable of yielding a result which is correct within these limits of error.

It was known in 1914 that the values accepted for the International Ohm at the P.T.R. and N.P.L. differed by 3×10^5, and that the results of the absolute measurements at the two laboratories differed by 4×10^5 and not 1×10^5 as a bare statement of the above results might suggest. On the other hand, it is also known that the values accepted for the International Ohm in 1936 at each of the Laboratories did not differ by as much as 1×10^5, so that the differences between the last three results are those between the absolute measurements. The N.P.L. values obtained in 1914 and 1936 are the same within the experimental error, and it is not possible to attach any significance to the difference between the corresponding two values given in Table VI.

The relation between the International Ohm and the ohm is shortly to be discussed by the Committees of the International Bureau of Weights and Measures, who will probably have additional data, and there would therefore be little point in attempting a full discussion at this juncture. The most significant facts arising from the work of the National Physical Laboratory are that it has been possible to determine an absolute unit of resistance by two different methods, with agreement within 1 part in 10^5, and that measurements made over a period of more than 20 years have agreed within the estimated limits of error, which are smaller than those involved in the establishment of the International Ohm as at present defined. With the Campbell method in particular all the readings can be obtained with a precision that is seldom equalled in any electrical measurements, and there can be no question that apparatus of this kind provides a far more satisfactory means of obtaining a unit of resistance than the standard mercury tube.

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